Slow-fast Bogdanov-Takens bifurcations in an application

P. De Maesschalck

June 2016

- This talk is about slow-fast systems

$$\begin{cases} \dot{x} &= f(x, y, \varepsilon, \lambda) \\ \dot{y} &= \varepsilon g(x, y, \varepsilon, \lambda) \end{cases}$$

where $\varepsilon > 0$ is small, λ is some parameter.

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- Mathematical foundations:
  - Geometric singular perturbation Theory by Fenichel, Jones
  - Desingularization by Dumortier, Roussarie
  - Canards by Benoit et al
  - Asymptotics by Eckhaus, Wasow, Ramis et al

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- Motivation is 2-fold:
  - Study of periodic orbits (Hilbert 16th problem)
  - Applications to natural rhythms in biology, neurology, ecology, ...

The Fitzhugh-Nagumo model and slow-fast Hopf bifurcations

$$\left\{\begin{array}{rcl} \dot{v} & = & v - \frac{1}{3}v^3 - w + I \\ \dot{w} & = & \varepsilon(v + a - bw) \end{array}\right.$$

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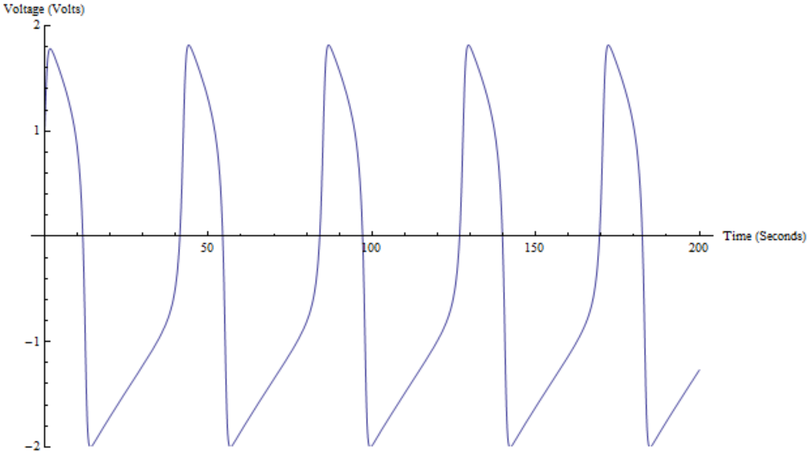

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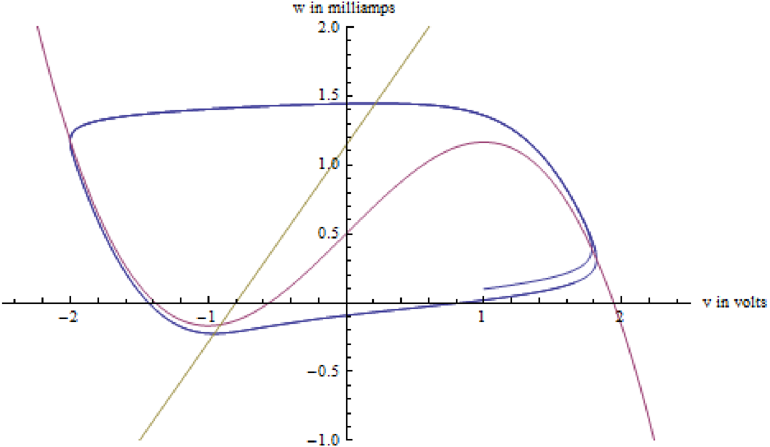
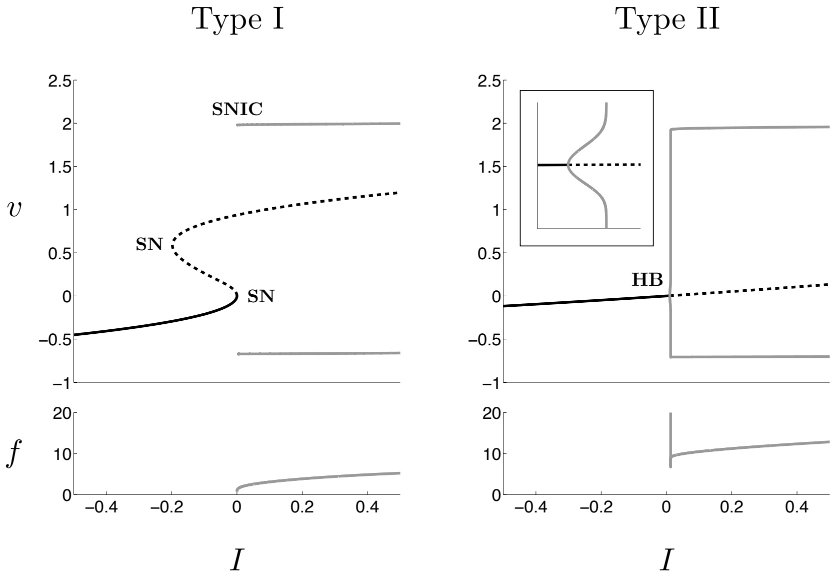
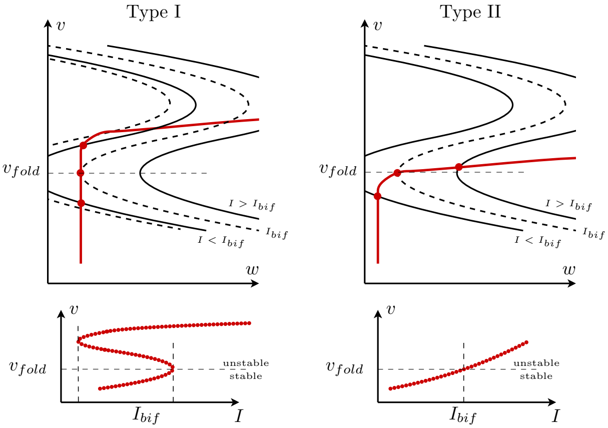


Type I vs type II excitation



Changing the w-nullcline



Slow-fast equations

$$\left\{ \begin{array}{lcl} w' & = & \varepsilon g(w, v, \varepsilon, \lambda) \\ v' & = & f(w, v, \varepsilon, I) \end{array} \right. \tag{1}$$

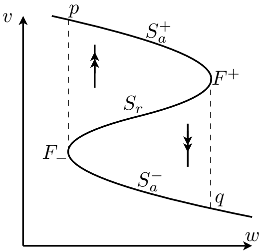
A more specific neuronal model:

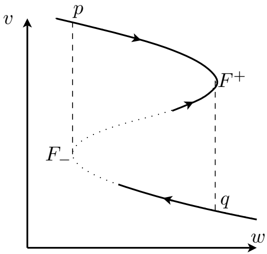
$$\begin{aligned} w' &= \varepsilon(G(v) - w) \\ v' &= v^2(d - v) - w + I\,, \end{aligned} \tag{2}$$

with

$$G(v) = \begin{cases} cv, & v \leq v_{\text{th}} \\ cv + e(v - v_{\text{th}})^2, & v > v_{\text{th}} \end{cases} \tag{3}$$

Slow-fast analysis



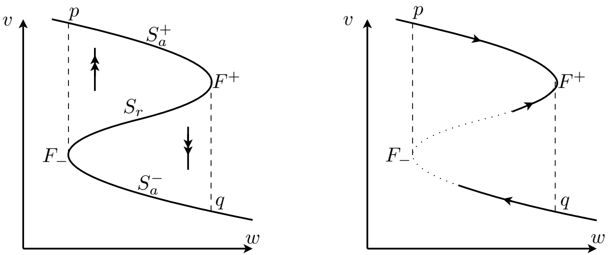


$$\begin{cases} w' &= 0 \\ v' &= v^2(d-v) - w + I \end{cases} \quad \begin{cases} w' &= G(v) - w \\ 0 &= v^2(d-v) - w + I \end{cases}$$

The critical manifold S is cubic shaped and given as a graph $\{w = \phi_I(v)\}$, i.e.

$$S = S_a^- \cup F^- \cup S_r \cup F^+ \cup S_a^+ \,,$$

Slow-fast analysis

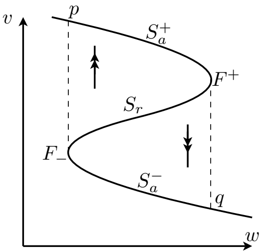


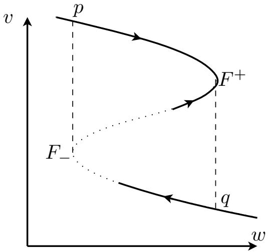
$$\begin{cases} w' = 0 \\ v' = v^2(d-v) - w + I \end{cases} \quad \begin{cases} w' = G(v) - w \\ 0 = v^2(d-v) - w + I \end{cases}$$

Along the w-nullcline $g(w, v, 0, \lambda) = 0$:

$$\frac{\partial g}{\partial w} \neq 0\,, \quad \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial w} \leq 0\,.$$

Slow-fast analysis

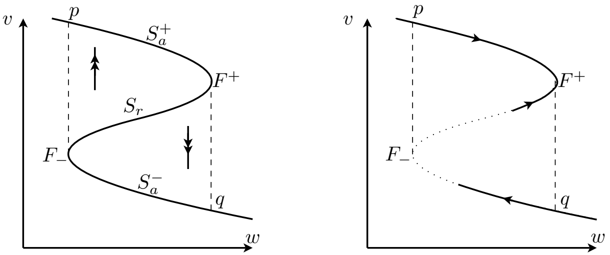




$$\begin{cases} w' &= 0 \\ v' &= v^2(d-v) - w + I \end{cases} \qquad \begin{cases} w' &= G(v) - w \\ 0 &= v^2(d-v) - w + I \end{cases}$$

The system can have one, two or three equilibria on $w = \phi_I(v)$, all of them located either on S_r or on S_a^-

Slow-fast analysis



$$\begin{cases} w' = 0 \\ v' = v^2(d-v) - w + I \end{cases} \qquad \begin{cases} w' = G(v) - w \\ 0 = v^2(d-v) - w + I \end{cases}$$

The fold point $F^+ = (w^+, v^+)$ is a regular jump point.

Assumption

Consider

$$\begin{cases} w' &= \varepsilon g(w, v, \varepsilon, \lambda) \\ v' &= f(w, v, \varepsilon, I) \end{cases}$$

and define

$$\mathcal{G}(v, I, \lambda) = g(\phi_I(v), v, 0, \lambda).$$

For fixed $(I, \lambda) = (I_{\text{bif}}, \lambda_{\text{bif}})$, the fold point $F^- = (w^-, v^-)$ is a singular contact point that undergoes a singular Bogdanov-Takens bifurcation with respect to the parameters (I, λ):

$$\mathcal{G}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) = 0, \quad \frac{\partial \mathcal{G}}{\partial v}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) = 0, \quad \frac{\partial^2 \mathcal{G}}{\partial v^2}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) > 0,$$

$$\frac{\partial \mathcal{G}}{\partial I}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) \neq 0, \quad \frac{\partial \mathcal{G}}{\partial \lambda}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) = 0, \quad \frac{\partial^2 \mathcal{G}}{\partial \lambda \partial v}(v^-, I_{\text{bif}}, \lambda_{\text{bif}}) \neq 0.$$

Besides the possible singular points near F^- occurring in this bifurcation, there are no other singular points on S_a^-.

Proposition

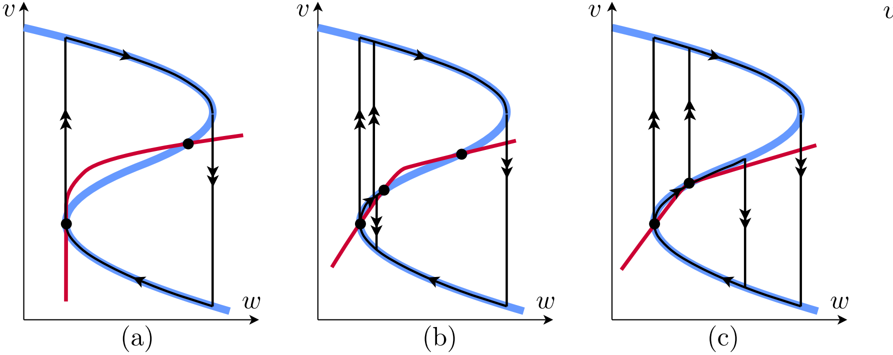
Under these assumptions, the family of vector fields can be locally transformed in the following normal form near F^-,

$$\begin{aligned} x' &= \varepsilon \left(cy - \sigma x - a + O(x^2, y^3, xy, \varepsilon y^2)\right) \\ y' &= y^2 - x + \beta y^3 + O(y^4)\,, \end{aligned} \tag{4}$$

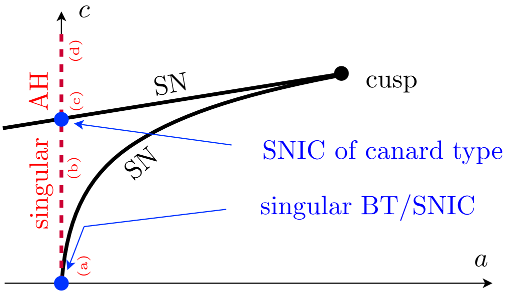
where $\sigma = \pm 1$, $\sigma c \geq 0$ and $\beta \neq 0$. The coefficients a, c and β can be computed explicitly in terms of $(I, \lambda, \varepsilon)$.

For system (2), the coefficients in the normal form are given by $c \geq 0$ and

$$a = dI, \qquad \beta = -1/d^2, \qquad \sigma = 1, \tag{5}$$



(a) $a = c = 0$: singular SNIC
(b) $a = 0$, $c > 0$ below c_{cusp}: slow-fast Hopf (truncated)
(c) $a = 0$, $c = c_{cusp}$: slow-fast Hopf (truncated)
(d) $a = 0$, $c > c_{cusp}$: slow-fast Hopf



c
(d)
AH
SN
cusp
(c)
SNIC of canard type
singular
(b)
SN
singular BT/SNIC
a
(a)

Theorem

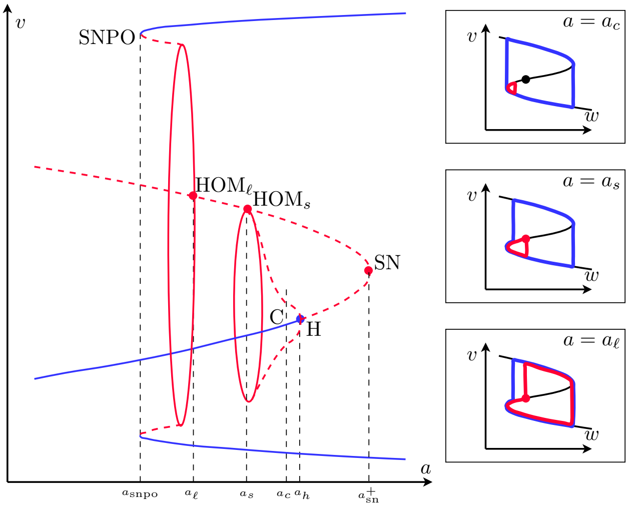
For fixed $0 < c < c_{cusp}$ and $0 < \varepsilon \ll 1$ there exists an unstable equilibrium on the middle branch $S_{r,\varepsilon}$ bounded away from the lower fold F^-. Furthermore, there exist functions

$$0 < a_{snpo}(\varepsilon) < a_\ell(\varepsilon) < a_s(\varepsilon) < a_c(\varepsilon) < a_h(\varepsilon) < a_{sn}^+(\varepsilon)$$

that all converge to zero in the singular limit $\varepsilon \to 0$ (except a_{sn}^+) and for which the following holds:

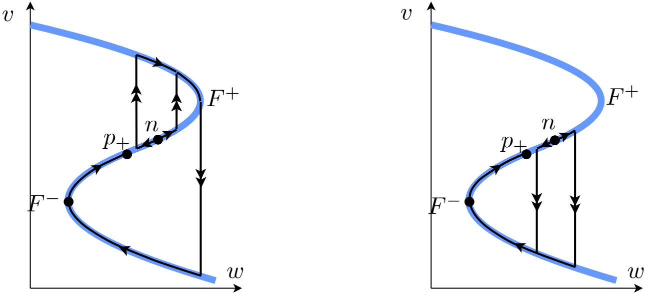
1. *For $a_{sn}^+ < a$, the fold F^- is of regular jump type and a large stable relaxation cycle exists.*
2. *At $a = a_{sn}^+$, a saddle-node bifurcation of singular points on the middle branch $S_{r,\varepsilon}$ in an $O(c)$-neighbourhood of F^-; the large relaxation cycle persists.*
3. *For $a_h < a < a_{sn}^+$, the system has a saddle p_+ and an unstable focus/node p_- on the middle branch $S_{r,\varepsilon}$ surrounded by the large relaxation cycle. The unstable focus/node p_- is closer to the fold F^-.*

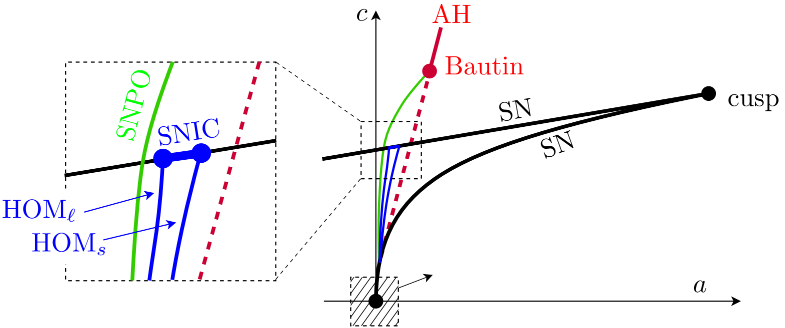
4. At $a = a_h$, p_- changes stability and a subcritical singular Andronov-Hopf bifurcation takes place; the large relaxation cycle persists.
5. For $a_c < a < a_h$, repelling small amplitude limit cycles appear around the stable focus p_-; the large relaxation cycle persists.
6. For $a_s < a < a_c$, small jump-back canard cycles appear that rapidly grow in amplitude (canard explosion); the large relaxation cycle perturbs to a large-amplitude jump-forward canard cycle.
7. At $a = a_s$, a small jump-back homoclinic loop of canard type, issued from the saddle p_+, appears together with a stable large-amplitude canard cycle.
8. For $a_\ell < a < a_s$, the small homoclinic loop breaks and only the stable large-amplitude canard cycle persists.
9. At $a = a_\ell$, a large-amplitude homoclinic loop of canard type, issued from the saddle p_+, appears together with the outer large-amplitude cycle.
10. As a decreases from a_ℓ, large amplitude canard cycles appear that grow in amplitude until it disappears in a saddle-node bifurcation of limit cycles at $a = a_{snpo}$



v
SNPO
HOM$_\ell$
HOM$_s$
SN
C
H
a
a_{snpo}
a_ℓ
a_s
a_c
a_h
a_{sn}^+
$a = a_c$
v
w
$a = a_s$
v
w
$a = a_\ell$
v
w

Heteroclinic connections of canard type undergo a transition from headless canard to canard with head, from the jump-back canard homoclinic to the jump-away canard homoclinic:





AH
Bautin
cusp
SN
SN
c
a
SNPO
SNIC
HOM$_\ell$
HOM$_s$

In order to get a hold on the parameters close to $c = 0$, we rescale the parameters and introduce

$$(c, a) = (\varepsilon C, \varepsilon^2 A), \qquad (C, A) \in [0, M] \times [-M, M] \tag{6}$$

for some large $M > 0$. By doing this we in fact assume that $c = O(\varepsilon)$ and $a = O(\varepsilon^2)$. After the parameter rescaling (6), we study the system

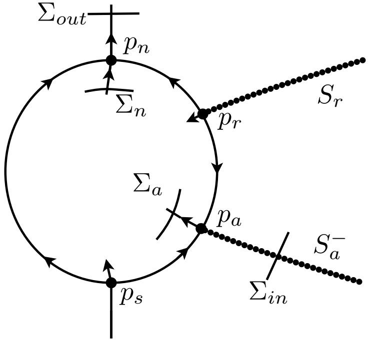
$$\begin{aligned} x' &= \varepsilon \left(-\varepsilon^2 A + \varepsilon C y - x + O(x^2, y^3, xy, \varepsilon y^2)\right) \\ y' &= y^2 - x + O(y^3) \,. \end{aligned} \tag{7}$$

The singularity at $(x, y, \varepsilon) = (0, 0, 0)$ is a *slow-fast Bogdanov-Takens point*.

Near the fold, we study the system using *blow-up* [**?**, **?**]. We write

$$(x, y, \varepsilon) = (r^2 X, rY, rE), \qquad r \geq 0, (X, Y, E) \in S^2_+$$

where S^2_+ denotes the half-sphere $X^2 + Y^2 + E^2 = 1$ with $E \geq 0$ (also known as *Poincaré or blow-up sphere*). The weights are chosen in a way that the higher order (big-oh) terms in (12) have also higher order in the rescaled equation.

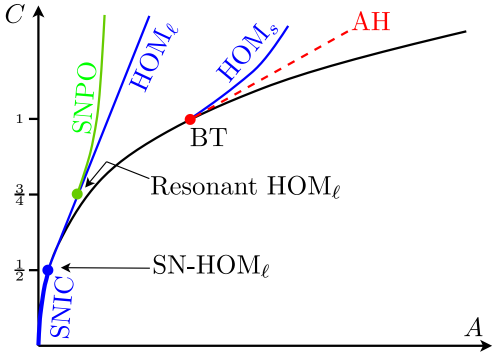


$$
\begin{aligned}
x' &= \varepsilon \left(cy - \sigma x - a + O(x^2, y^3, xy, \varepsilon y^2) \right) \\
y' &= y^2 - x + \beta y^3 + O(y^4) \, ,
\end{aligned} \tag{8}
$$

$$
(x, y) = (\varepsilon^2 X, \varepsilon Y), \qquad (X, Y) \in [-R, R]^2, \tag{9}
$$

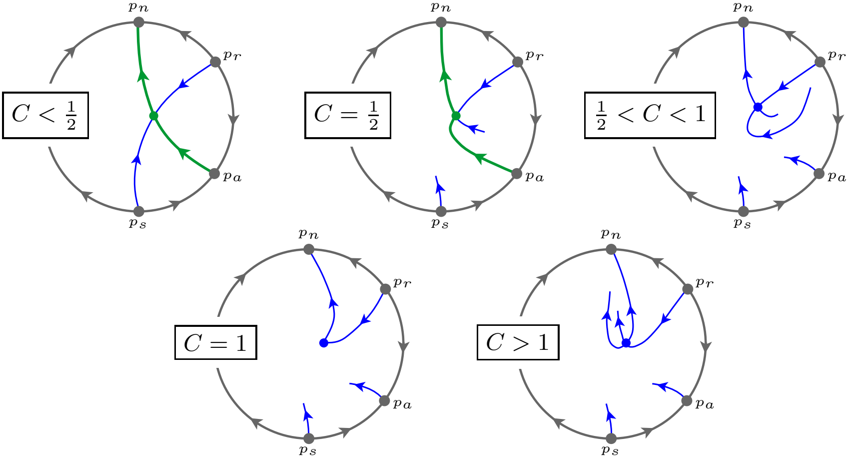
for some large $R > 0$. Applying this rescaling to (12), we can divide out a common factor ε, thus transforming the system into a regular perturbation family

$$
\begin{aligned}
\dot{X} &= -A + CY - X + O(\varepsilon), \\
\dot{Y} &= Y^2 - X + O(\varepsilon).
\end{aligned} \tag{10}
$$



C
A
1
$\frac{3}{4}$
$\frac{1}{2}$
SNPO
HOM$_\ell$
HOM$_s$
AH
BT
Resonant HOM$_\ell$
SN-HOM$_\ell$
SNIC

$A = C^2/4$:



Theorem

There exists a parameter surface $A^+_{sn}(C, \varepsilon) = C^2/4 + O(\varepsilon)$ *along which a saddle-node singularity* $p_\pm$ *exists. On this surface, there exists a curve* $C = \frac{1}{2} + O(\varepsilon)$ *along which a saddle-node homoclinic (SN-HOM*$_\ell$*) connection appears containing the hyperbolic separatrix of the saddle-node. For* $C < \frac{1}{2} + O(\varepsilon)$ *on this parameter surface, there is a SNIC connection containing a center separatrix of the saddle-node. For* $C > \frac{1}{2} + O(\varepsilon)$*, there is no SNIC connection.*

Proof: SN-bifurcation is stable so there exists A^+_{sn}-curve which is perturbation of $A = C^2/4$.
There exists a C^k- center outgoing separatrix W and a C^∞ incoming stable separatrix V.
Both are (C, ε)-families of curves. Intersect V with a transverse section parameterized by a coordinate s so that

$$V\colon s = \psi(C, \varepsilon),$$

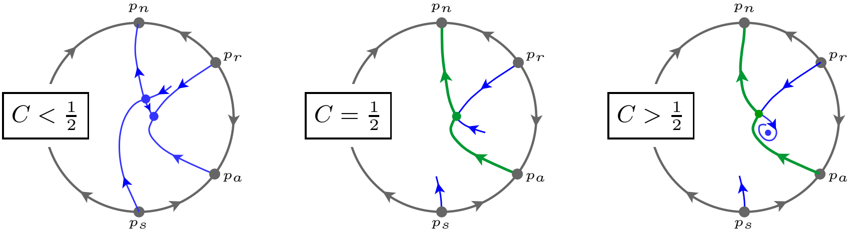
for some smooth ψ. Then integrate W following the vector field until it reaches V. This gives

$$s = \phi(C, \varepsilon),$$

for some C^k-function ϕ.
Next $\phi(1/2, 0) = \psi(1/2, 0)$ and $\frac{\partial \phi}{\partial C}(1/2, 0) \neq \frac{\partial \psi}{\partial C}(1/2, 0)$ (Melnikov-like computation).

So there we can apply IFT: there exists $C = C(\varepsilon)$ along which a SN-HOM connection appears.
Finally for $C < C(\varepsilon)$ we apply the technique of rotating vector fields to see that the SNIC connection is made.
The same technique shows that $C > C(\varepsilon)$ shows a big relaxation cycle.

$$A = -\frac{1}{16} + \frac{C}{4}:$$



Theorem

Let $C_{min} > \frac{1}{2}$. There exists a parameter surface $A_\ell(C, \varepsilon) = -\frac{1}{16} + \frac{C}{4} + O(\varepsilon)$, $C > C_{min}$ along which a large-amplitude saddle-homoclinic (HOM_ℓ) connection exists. On this surface, there exists a curve $C = \frac{3}{4} + O(\varepsilon)$ along which the homoclinic changes stability (Resonant HOM_ℓ): for lower values of C, the homoclinic is stable, for larger values it is unstable. From this curve emerges a surface $A = A_{snpo}(C, \varepsilon)$ along which a SNPO bifurcation takes place. The surfaces $A_{snpo}(C, \varepsilon)$ and $A_\ell(C, \varepsilon)$ are exponentially close.

Proof:

The existence is HOMOCLINIC is similar to that of SN-HOM. (A little attention must be paid gluing HOM to SN-HOM.)

Take C^1-normal form around hyperbolic saddle (for equivalence)

$$\left\{ \begin{array}{lcl} \dot{X} & = & -X \\ \dot{Y} & = & \rho(C)Y, \qquad \rho(C,\varepsilon) := 4C - 2 + O(\varepsilon) > 0. \end{array} \right.$$

Take a section $Y = 1$ with coordinate x_0 and integrate backward in time until we reach the section $X = 1$ with coordinate y_0. This gives

$$y_0 = x_0^{\rho(C,\varepsilon)}.$$

Next integrate in backward time.

$$y_0 = \phi(A, C, \varepsilon) + \exp(-\tilde{I}(\varepsilon^2 x_0, A, C, \varepsilon)/\varepsilon).$$

(formula explained on blackboard)

Along $A=-\frac{1}{16}+\frac{C}{4}$ we have a HOM connection given by $x_0=0$, so

$$\phi(A,C,\varepsilon)+\exp(-\tilde{I}(0,A,C,\varepsilon)/\varepsilon)=0.$$

Applying again a Melnikov argument shows that this bifurcation line perturbs to $\varepsilon>0$.

The Saddle-node of Periodic Orbits (SNPO):

$$\Delta:=\phi(A,C,\varepsilon)+\exp(-\tilde{I}(\varepsilon^2 x_0,A,C,\varepsilon)/\varepsilon)-x_0^{\rho(C,\varepsilon)}.$$

We derive with respect to x_0:

$$\Delta'=\exp(-\hat{I}(\varepsilon^2 x_0,A,C,\varepsilon)/\varepsilon)-\rho(C,\varepsilon)x_0^{\rho(C,\varepsilon)-1}.$$

It is clear that when $\rho(C)<1$ this expression tends to $-\infty$ as $x_0\to 0$. When $\rho(C)>1$ we find a critical point at a solution of the equation

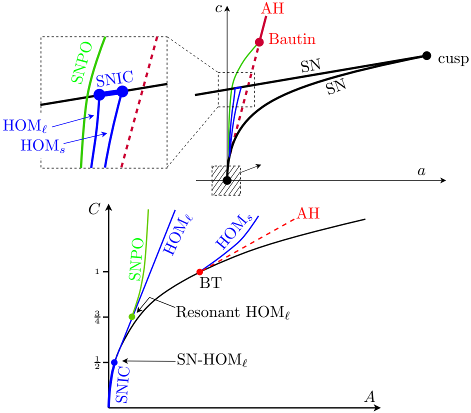
$$x_0=\left(\frac{1}{\rho}\right)^{1-\rho}\exp\left(\frac{-\hat{I}(\varepsilon^2 x_0,A,C,\varepsilon)}{\varepsilon(1-\rho)}\right).$$

Clearly, this solution is of exponentially-flat type (w.r.t. ε and $1-\rho$). Combine with

$$\phi(A, C, \varepsilon) + \exp(-\tilde{I}(\varepsilon^2 x_0, A, C, \varepsilon)/\varepsilon) = x_0^{\rho(C,\varepsilon)}$$

to find $A = A_{snpo}(C), \varepsilon)$.

How to connect both diagrams:



Before we have used the rescaling

$$(c, a) = (\varepsilon C, \varepsilon^2 A), \qquad (C, A) \in [0, M] \times [-M, M]$$

It is better to do

$$(c, a, \varepsilon) = (vC, v^2 A, vE), \qquad (C, A, E) \in S^2, v \geq 0.$$

The previous rescaling amounts to looking on a chart of the sphere in the direction of $E = 1$.

$$(c, a, \varepsilon) = (v, v^2 A, vE), \qquad E \approx 0, A \in [-M, M], v \geq 0.$$

Since $c = v$ we can simplify to

$$(a, \varepsilon) = (c^2 A, cE), \qquad E \approx 0, A \in [-M, M], c \geq 0.$$

The vector field:

$$\begin{aligned} x' &= cE\left(-c^2 A + cy - x + O(x^2, y^3, xy, \varepsilon y^2)\right) \\ y' &= y^2 - x + O(y^3)\,. \end{aligned} \tag{11}$$

$$
\begin{aligned}
x' &= cE\left(-c^2A + cy - x + O(x^2, y^3, xy, \varepsilon y^2)\right) \\
y' &= y^2 - x + O(y^3)\,.
\end{aligned} \tag{12}
$$

$$
(x, y) = (c^2X, cY), \qquad (X, Y) \in [-R, R]^2, \tag{13}
$$

for some large $R > 0$. Applying this rescaling to (12), we can divide out a common factor ε, thus transforming the system into a regular perturbation family

$$
\begin{aligned}
\dot{X} &= E(-A + Y - X + O(c)), \\
\dot{Y} &= Y^2 - X + O(c).
\end{aligned}
$$

$\Longrightarrow$ in this parameter regime the Bogdanov-Takens contact point blows up to a slow-fast Hopf situation.

$$\dot{X} = E(-A + Y - X + O(c)),$$
$$\dot{Y} = Y^2 - X + O(c).$$

We find at $A = 1/4 + O(c)$ a saddle-node in the slow dynamics, at $A = 0 + O(c)$ a slow-fast Hopf point.
At the slow-fast Hopf point there is an extra singularity on the middle branch, giving rise to HOM-connections.
$\implies$ the bifurcation diagram is complete!