A mixed finite element method for Phan-Thien-Tanner differential model

Reporter： Yunqing Feng
Supervisor： Lei Hou

2016 International Conference on Singular Perturbation Theory and its Applications

2016.06.26

Outline

1. The background of research
2. IP method for stokes-like problem
3. SUPG method for constitutive equation
4. Numerical experiments

Outline

1. The background of research
2. IP method for stokes-like problem
3. SUPG method for constitutive equation
4. Numerical experiments

The background of research

- In recent years, there have been a lot of developments for solving viscoelastic flows. In this report, we solve viscoelastic flows with Phan-Thien-Tanner(P-T-T) model, which consists of the momentum equation, mass equation and constitutive equation.

P-T-T model

$$-\nabla \cdot \boldsymbol{T} + \nabla p = \boldsymbol{f}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau} + \lambda((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} - \boldsymbol{\tau}\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^T\boldsymbol{\tau}) = 2(1-\eta_p)\mathbf{D}(\boldsymbol{u}), \tag{3}$$

where $\boldsymbol{u}$ and p denote the velocity and pressure fields, respectively, ε , λ and η_p denote the constants. $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ denote the rate of deformation tensor. The relationship between the extra stress tensor $\boldsymbol{T}$ and viscoelastic stress tensor $\boldsymbol{\tau}$ can be defined as following

$$\boldsymbol{T} = 2\eta_p\mathbf{D}(\boldsymbol{u}) + \boldsymbol{\tau}. \tag{4}$$

- There are lots of work devoted to the numerical approximations for the viscoelastic flows. For example, Least-squares finite element method, Discntinuous Galerkin method and so on.
- In this report, we consider a numerical method for the P-T-T model of viscoelastic flows by a combination of the interior penalty (IP) method and the streamline upwind Petrov-Galerkin (SUPG) method.

- Recently, the interior penalty(IP) method for convection-diffusion problem was proposed by Douglas and Dupont. We extend Bonito and Burman's work in [1] to a finite element scheme of P-T-T model and derive the error estimates for the numerical solution.
- We decouple the P-T-T model into two parts: the stokes-like problem and the constitutive equation. We consider an IP method for the stokes-like equations and adopt SUPG method to discretize the constitutive equation.

Outline

1 The background of research

2 IP method for stokes-like problem

3 SUPG method for constitutive equation

4 Numerical experiments

Stokes-like problem

$$\begin{cases} -\nabla \cdot \boldsymbol{T} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{T} = 2\eta_p \mathbf{D}(\boldsymbol{u}) + \boldsymbol{\tau}, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

we introduce the bilinear forms

$$a_h(\boldsymbol{T}_h, \boldsymbol{v}_h) = (\boldsymbol{T}_h, \mathbf{D}(\boldsymbol{v}_h)) - \langle \boldsymbol{T}_h \cdot \boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial\Omega}, \tag{6}$$

$$b_h(p_h, \boldsymbol{v}_h) = -(p_h, \nabla \cdot \boldsymbol{v}_h) + \langle p_h, \boldsymbol{v}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega}, \tag{7}$$

Stokes-like problem

and the interior penalty operators

$$j_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = 2\eta_p \sum_{K \in \varepsilon_h^0} h \int_{\partial K} [\nabla \boldsymbol{u}_h] \cdot [\nabla \boldsymbol{v}_h] \mathrm{d}s + \frac{\eta_p \beta}{h} \int_{\partial \Omega} \boldsymbol{u}_h \cdot \boldsymbol{v}_h \mathrm{d}s, \quad (8)$$

$$j_2(p_h, q_h) = \sum_{K \in \varepsilon_h^0} \frac{\gamma h^3}{\eta_p} \int_{\partial K} [\nabla p_h][\nabla q_h] \mathrm{d}s, \quad (9)$$

where α, β, γ are positive constants.

Stokes-like problem

The weak formulation of problem (5) reads as follows: find $(\boldsymbol{u}_h, p_h, \boldsymbol{T}_h) \in X_h$ such that

$$a_h(\boldsymbol{T}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}_h) - a_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) + \left(\frac{1}{2\eta_p}\boldsymbol{T}_h, \boldsymbol{\sigma}_h\right)$$
$$+ j_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + j_2(p_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{\tau}, \boldsymbol{\sigma}_h)\,, \tag{10}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\sigma}_h) \in X_h$.

Stokes-like problem

Introducing the variables $M_h = (\boldsymbol{u}_h, p_h, \boldsymbol{T}_h)$ and $N_h = (\boldsymbol{v}_h, q_h, \boldsymbol{\sigma}_h)$, the finite element formulation (10) can be written as $A(M_h, N_h) = a_h(\boldsymbol{T}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}_h) - a_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) + \left(\frac{1}{2\eta_p}\boldsymbol{T}_h, \boldsymbol{\sigma}_h\right)$, and $J(M_h, N_h) = j_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + j_2(p_h, q_h)$, $F(N_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{\tau}, \boldsymbol{\sigma}_h)$. That is, we find $M_h \in X_h$ such that

$$A(M_h, N_h) + J(M_h, N_h) = F(N_h), \qquad \forall N_h \in X_h \tag{11}$$

Theorem 1

Theorem

Suppose that the mesh satisfies the quasiuniformity of the mesh and that M be the solution of (5), then the solution M_h by the interior penalty method satisfies the error estimate

$$||| M - M_h |||\leq C \| \boldsymbol{\tau} - \boldsymbol{\tau}_h \| + o(h), \tag{12}$$

where C is a constant independent of h.

Outline

1. The background of research
2. IP method for stokes-like problem
3. SUPG method for constitutive equation
4. Numerical experiments

We define

$$B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\omega}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau},\boldsymbol{\omega}+h(\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}) + \frac{1}{2}((\nabla\cdot\boldsymbol{u})\boldsymbol{\tau},\boldsymbol{\omega}), \tag{13}$$

for all $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\omega}) \in V_h \times V_h \times Z_h \times Z_h$. Moreover, setting $\boldsymbol{u}=\boldsymbol{v}, \boldsymbol{\tau}=\boldsymbol{\omega}$, we have

$$B(\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\tau}) = h((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau},(\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}) = h \mid\mid (\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} \mid\mid^2 . \tag{14}$$

For $\boldsymbol{\omega}_u = \boldsymbol{\omega} + \upsilon h\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}$, we obtain

$$B(\lambda\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\omega}) = ((\lambda\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau},\boldsymbol{\omega}+\upsilon h(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}) + \frac{1}{2}((\nabla\cdot\lambda\boldsymbol{u})\boldsymbol{\tau},\boldsymbol{\omega}). \tag{15}$$

Now we define the discrete approximation of (3) as, find $\boldsymbol{\tau}\in Z_h$ such that

$$\begin{aligned}&\left(\left(1+\frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h,\boldsymbol{\omega}_{u_h}\right) + B(\lambda\boldsymbol{u}_h,\boldsymbol{\tau}_h,\boldsymbol{\omega})\\ &-\lambda((\boldsymbol{\tau}_h\nabla\boldsymbol{u}_h+\nabla\boldsymbol{u}_h^T\boldsymbol{\tau}_h),\boldsymbol{\omega}_{u_h}) = 2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}_h),\boldsymbol{\omega}_{u_h}), \quad \forall\boldsymbol{\omega}\in Z_h.\end{aligned} \tag{16}$$

Lemma 2

Let

$$L = \max\{\| \boldsymbol{u} \|_3, \| p \|_2, \| \boldsymbol{T} \|_2, \| \boldsymbol{\tau} \|_2\}$$

and

$$\max\{\| \nabla \boldsymbol{u}_h \|_{0,\infty}, \| \boldsymbol{\tau}_h \|_{0,\infty}\} \leq M.$$

Utilizing the above hypotheses, the prior error estimate can be derived as follow.

Lemma

Let $\boldsymbol{\tau}_h$ be a numerical solution of (3). Assuming that the exact solution $(\boldsymbol{u}, p, \boldsymbol{T}, \boldsymbol{\tau})$ is smooth enough. If C is a positive constant independent of h, then the following inequality holds

$$\| \boldsymbol{\tau} - \boldsymbol{\tau}_h \| \leq Ch^{3/2} + (2(1-\eta_p) + C\lambda L + C\lambda M h) \| \boldsymbol{u} - \boldsymbol{u}_h \|_1 /(7/8 - 2\lambda M - C\rho$$

for sufficiently small $\lambda > 0$ and $\rho = \frac{\varepsilon\lambda}{1-\eta_p}$.

Theorem 3

Theorem

Suppose that the exact solution $(\boldsymbol{u}, p, \boldsymbol{T}, \boldsymbol{\tau}) \in X \times Z$ *is smooth enough, then the finite element solution* $(\boldsymbol{u}_h, p_h, \boldsymbol{T}_h, \boldsymbol{\tau}_h) \in X_h \times Z_h$ *by the IP/SUPG method satisfies the following error bound, that is*

$$||| \, (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \boldsymbol{T} - \boldsymbol{T}_h) \, ||| + \| \, \boldsymbol{\tau} - \boldsymbol{\tau}_h \, \| \leq Ch. \tag{17}$$

Proof.

Combining Theorem 1 with Lemma 2, we obtain the bound (17), which completes the proof. □

Numerical experiments

In this part, numerical results of the creeping flow in a planar channel are presented by the above decoupled algorithm.We choose the exact solution $(\boldsymbol{u}, p, \boldsymbol{\tau})$

$$\begin{aligned} \boldsymbol{u} &= \begin{pmatrix} 1 - y^4 \\ 0 \end{pmatrix}, \\ p &= -x^2, \\ \boldsymbol{\tau} &= \begin{pmatrix} 32\lambda\alpha y^6 & -4\alpha y^3 \\ -4\alpha y^3 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Therefore, the right hand side terms of the momentum and constitutive equations modified by substituting (18) into (5) and (3) are given by

$$\boldsymbol{f} = \begin{pmatrix} 12y^2 - 2x \\ 0 \end{pmatrix}, \tag{19}$$

and

$$\boldsymbol{f}_{stress} = \begin{pmatrix} 32\lambda y^6/9 + 1024\varepsilon\lambda^3 y^{12}/9 & 128\lambda^2(1-\varepsilon)y^9/9 \\ 128\lambda^2(1-\varepsilon)y^9/9 & -32\lambda y^6/9 \end{pmatrix}. \tag{20}$$

The test domain with Dirichlet boundary conditions are shown in Figure 1. u_y represent the velocity in y-direction.

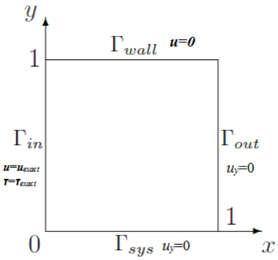

Figure: The geometry and boundary conditions.

The effects of material parameter λ on the error estimates of velocity, pressure and stress are displayed in Table 1 and Table 2.

Table 1. The error estimates and the convergence orders k of $\boldsymbol{u}$, p and $\boldsymbol{\tau}$ with $\lambda = 0.5$ by IP/SUPG algorithm.

1/h	L_2 Error-$\boldsymbol{u}$	k	L_2 Error-p	k	L_2 Error-$\boldsymbol{\tau}$	k
16	4.51×10^{-3}	-	5.30×10^{-2}	-	1.19×10^{-2}	-
32	1.26×10^{-3}	1.84	2.06×10^{-2}	1.36	3.95×10^{-3}	1.59
64	3.21×10^{-4}	1.97	7.35×10^{-3}	1.49	1.17×10^{-3}	1.76

Table 2. The error estimates and the convergence orders k of $\boldsymbol{u}$, p and $\boldsymbol{\tau}$ with $\lambda = 2.1$ by IP/SUPG algorithm.

1/h	L_2 Error-$\boldsymbol{u}$	k	L_2 Error-p	k	L_2 Error-$\boldsymbol{\tau}$	k
16	4.82×10^{-3}	-	5.52×10^{-2}	-	1.26×10^{-2}	-
32	1.39×10^{-3}	1.79	2.18×10^{-2}	1.34	4.42×10^{-3}	1.51
64	3.71×10^{-4}	1.91	8.21×10^{-3}	1.41	1.39×10^{-3}	1.67

References

A. Bonito and E. Burman, *A Continuous Interior Penalty Method for Viscoelastic Flows*, Siam Journal on Scientific Computing, 38 (2008), pp. 1156-1177.

S. Zhou and L. Hou, *Decoupled algorithm for solving Phan-Thien-Tanner viscoelastic fluid by finite element method*, Computers and Mathematics with Applications, 69 (2015), pp. 423-437.

A. Bonito and E. Burman, *A Face Penalty Method for the Three Fields Stokes Equation Arising from Oldroyd-B Viscoelastic Flows*, Numerical Mathematics and Advanced Applications, 2005, pp. 487-494.

Thank you!