A GEOMETRIC APPROACH TO STATIONARY DEFECT SOLUTIONS IN ONE SPACE DIMENSION

(joint work with A. Doelman and P. van Heijster, SIADS, 2016)

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Overview

We formulate a general theory on the effect of small jump-like defects (which we will call *weak defects*) in discontinuous inhomogeneous nonautonomous systems of ODEs:

$$\dot{u} = \begin{cases} f(u), & t \leq 0, \\ f(u) + \varepsilon g(u), & t > 0, \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^n$, $f(u), g(u) : \mathbb{R}^n \to \mathbb{R}^n$ are sufficiently smooth, and ε is a small positive parameter.

[cf. D.W. McLaughlin and A.C. Scott, Perturbation analysis of fluxon dynamics, Phys. Rev. A, 1978.]

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The perturbed generalized three-component FitzHugh-Nagumo system [P. van Heijster, etc, Nonlinearity 2011]

$$\begin{aligned} U_t &= \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma(x)) \\ \tau V_t &= V_{xx} + U - V \\ \theta W_t &= D^2 W_{xx} + U - W\,, \end{aligned} \tag{1.2}$$

with $(x,t) \in (\mathbb{R}, \mathbb{R}^+), \alpha, \beta \in \mathbb{R}, D > 1, \tau, \theta > 0, 0 < \varepsilon \ll 1$, and

$$\gamma(\xi) = \begin{cases} \gamma_1 & \text{for } x \leq 0\,, \\ \gamma_2 & \text{for } x > 0\,, \end{cases} \tag{1.3}$$

where $\gamma_{1,2} \in \mathbb{R}$.

Written as a system of six first order ODEs

$$\begin{cases} u_\xi = p\,, \\ p_\xi = -u + u^3 + \varepsilon(\alpha v + \beta w + \gamma(\xi))\,, \\ v_\xi = \varepsilon q\,, \\ q_\xi = \varepsilon(v-u)\,, \\ w_\xi = \frac{\varepsilon}{D} r\,, \\ r_\xi = \frac{\varepsilon}{D}(w-u)\,, \end{cases} \tag{1.4}$$

with $\xi := x/\varepsilon, \alpha, \beta \in \mathbb{R}, D > 1, 0 < \varepsilon \ll 1$, and $\gamma(\xi)$ as in (1.3).

- The original homogeneous three-component version: to explore gas discharge phenomena [M. Bode, A.W. Liehr, C.P. Schenk and H.-G. Purwins, Physica D 2002]
- Stable pinned stationary front and pulse solutions with the front or back of the solution located near the weak defect, *i.e.* near $x = 0$ [P. van Heijster, etc, Nonlinearity 2011]
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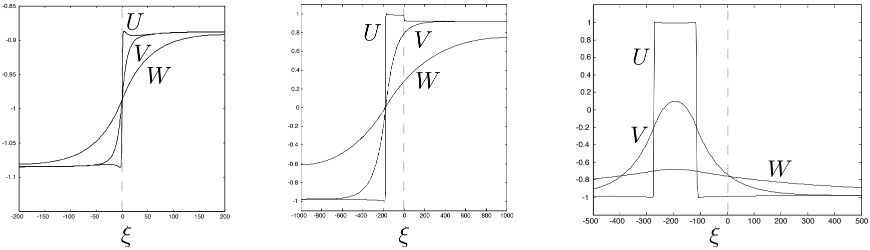

Figure 1.1: Left panel: a trivial weak defect solution supported by (1.2). The system parameters were as follows $(\alpha, \beta, D, \gamma_1, \gamma_2, \tau, \theta, \varepsilon) = (3, -2, 5, 3, -1, 1, 1, 0.1)$. Middle panel: a local weak defect solution in the shape of a stationary front solution supported by (1.2). The system parameters were as follows $(\alpha, \beta, D, \gamma_1, \gamma_2, \tau, \theta, \varepsilon) = (3, 2, 5, 0, 10, 1, 1, 0.01)$. Right panel: A local weak defect solution in the shape of a stationary pulse solution supported by (1.2). The system parameters were as follows $(\alpha, \beta, D, \gamma_1, \gamma_2, \tau, \theta, \varepsilon) = (3, 2, 5, 2, 1, 1, 1, 0.01)$ (note that this panel is adapted from Figure 5 of [vHDKNU,2011]). The location of the defect is indicated by the dashed line and $\xi := x/\varepsilon$.

This leads to the following question: can we develop a general theory for the persistence and/or existence of defect solutions supported by (1.1) for generic perturbations $\varepsilon g(u)$ under mild, generic assumptions on the unperturbed system

$$\dot{u} = f(u), \quad u(t) : \mathbb{R} \to \mathbb{R}^n?$$

CONTENTS

- The unperturbed system, that is (1.1) with $\varepsilon = 0$, is homogeneous and continuous

$$\dot{u} = f(u), \quad t \in \mathbb{R}. \tag{2.1}$$

- Hypothesis (H1). *System (2.1) has N isolated equilibrium points P_i $(i = 1, 2, \cdots, N)$, where N is a positive integer or $+\infty$.*
- *The continuous perturbed system, that is,*

$$\dot{u} = f(u) + \varepsilon g(u), \quad t \in \mathbb{R}, \tag{2.2}$$

has N equilibrium points P_i^ε with $P_i^0 = P_i$ for $i = 1, 2, \cdots, N$, provided that ε is small enough.

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- We are interested in solutions to (1.1) that asymptote to a hyperbolic P^- at $-\infty$ and to a hyperbolic P_ε^+ at ∞, where $P^- \in \{P_1, P_2, \cdots, P_N\}$ and $P_\varepsilon^+ \in \{P_1^\varepsilon, P_2^\varepsilon, \cdots, P_N^\varepsilon\}$, under the assumption that the unperturbed system (2.1) supports an *isolated* heteroclinic orbit.
- Hypothesis (H2). *The unperturbed system (2.1) supports an heteroclinic orbit Γ connecting P^- with $P^+ = P_0^+$ in forward time.*

  More specifically, there is a $\Gamma \in \mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)$ and we assume that the intersection is "minimally non-transversal". That is, if $\dim(\mathcal{W}^u(P^-)) + \dim(\mathcal{W}^s(P^+)) \leq n$, then $\dim(\mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)) = 1$ and if $\dim(\mathcal{W}^u(P^-)) + \dim(\mathcal{W}^s(P^+)) = m > n$, then $\dim(\mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)) = m - n$.

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Definition 2.1

Assume (H1) and (H2) hold and assume that ε is sufficiently small. If $\mathcal{W}^{s,u}(P^+) \neq n$, then we call the perturbation (1.1) *of* (2.1) *a* **generic perturbation** *if the distance between $\mathcal{W}^{s,u}_{\mathrm{loc}}(P^+)$ and P^+_ε is strictly of order ε and not smaller, that is,*

$$d(\mathcal{W}^{s,u}_{\mathrm{loc}}(P^+), P^+_\varepsilon) = \mathcal{O}_s(\varepsilon)\,,$$

where $d(\cdot,\cdot)$ denotes the Euclidian distance.

Hypothesis (H3). *The perturbation* (1.1) *of* (2.1) *is a generic perturbation.*

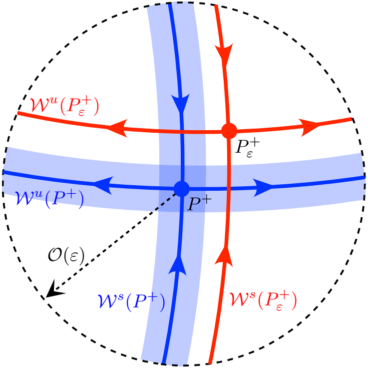

Figure 2.1: For a generic perturbation we have that $d(\mathcal{W}^{s,u}_{\text{loc}}(P^+), P_\varepsilon^+) = \mathcal{O}_s(\varepsilon)$. That is, P_ε^+ does not lie inside of the shaded blue regions. Moreover, since the perturbation is $\mathcal{O}(\varepsilon)$, the stable and unstable manifolds of P_ε^+ and P^+ are locally and to leading order *parallel.*

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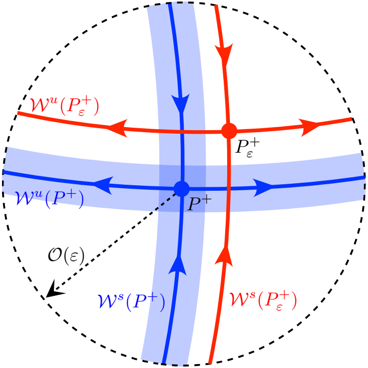

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Definition 2.2

A C^0-solution $\Gamma_\varepsilon(t)$ of (1.1) is called a **defect solution** *if*

$$\lim_{t\to-\infty}\Gamma_\varepsilon(t)=P^- \quad \textit{and} \quad \lim_{t\to+\infty}\Gamma_\varepsilon(t)=P^+_\varepsilon.$$

We distinguish between three types of defect solutions.

Definition 2.3

A defect solution $\Gamma_\varepsilon(t)$ is said to be a **trivial defect solution** *if $P^-=P^+$ and*

$$\lim_{\varepsilon\to 0}\|\Gamma_\varepsilon(t)-P^+_\varepsilon\|_\infty=0.$$

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Definition 2.4

A nontrivial weak defect solution $\Gamma_\varepsilon(t)$ *is said to be a* **local defect solution** *if either*

$$\lim_{\varepsilon\to 0}\|\Gamma_\varepsilon(t)-P_\varepsilon^+\|_{\infty,\mathbb{R}^+}=0, \quad \text{or} \quad \lim_{\varepsilon\to 0}\|\Gamma_\varepsilon(t)-P^-\|_{\infty,\mathbb{R}^-}=0, \tag{2.3}$$

where $\|\cdot\|_{\infty,\mathbb{R}^\pm}$ *denotes the* $\mathbb{L}^\infty$*-norm over* $\mathbb{R}^\pm$. *Moreover, we say that the defect occurs near* P_ε^+ *if the first condition of (2.3) holds and the defect occurs near* P^- *if the second condition holds.*

Finally, a nontrivial defect solution $\Gamma_\varepsilon(t)$ *is said to be a* **global defect solution** *if*

$$\lim_{\varepsilon\to 0}\|\Gamma_\varepsilon(t)-P_\varepsilon^+\|_{\infty,\mathbb{R}^+}>0, \quad \text{and} \quad \lim_{\varepsilon\to 0}\|\Gamma_\varepsilon(t)-P^-\|_{\infty,\mathbb{R}^-}>0.$$

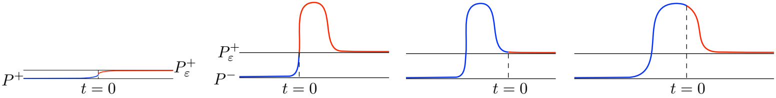

Figure 2.2: Left panel: trivial defect solution connecting P^+ with P_ε^+. Middle panels: local defect solution near P^-, P_ε^+, respectively. Right panel: global defect solution connecting P^- with P_ε^+.

Global defects in a perturbed stationary Fisher-KPP equation

Consider the following perturbed planar ODE

$$\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} = \begin{cases} \begin{pmatrix} p \\ u - u^2 \end{pmatrix}, & t \leq 0 \\ \begin{pmatrix} p + \varepsilon g_1(u,p) \\ u - u^2 + \varepsilon g_2(u,p) \end{pmatrix}, & t > 0 \end{cases} \tag{3.1}$$

where g_1 and g_2 are sufficiently smooth functions and ε is a small parameter.

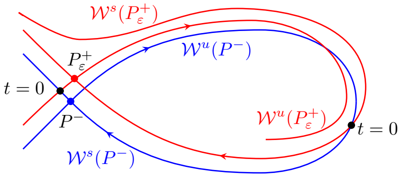

Figure 3.1: A typical sketch of the stable and unstable manifold of P^- (blue) and the stable and unstable manifold of P_ε^+ (red) from (3.1) in the case that $P^+ = P^-$, *i.e.* Γ of (H2) is actually an homoclinic orbit. There are two intersection points of $\mathcal{W}^u(P^-)$ and $\mathcal{W}^s(P_\varepsilon^+)$ (indicated by the black dots), which correspond to a trivial defect solution (the one closest to the equilibrium points) and a global defect solution, respectively.

- Global defect solutions are too hard to study in the general case.
- Hamiltonian structure. For example, for the non-generic perturbation $g_1 = 0$ and $g_2 = -u + 2u^2$, system (3.1) has two global defect solutions. This can be observed from the fact that

$$\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ u - u^2 + \varepsilon(2u^2 - u) \end{pmatrix},$$

still has an homoclinic orbit to $(0, 0)$, which has exactly two intersection points with the homoclinic orbit of (3.1) with $\varepsilon = 0$.

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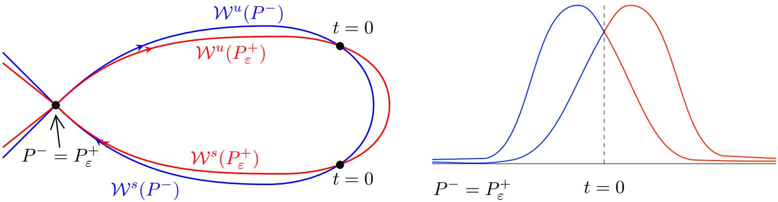

Figure 3.2: Left panel: the stable and unstable manifold of P^- and the stable and unstable manifold of P_ε^+ of (3.1) with $g_1 = 0$ and $g_2 = -u + 2u^2$. The two intersection points correspond to two global defect solutions which are shown in the right panel.

- $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+))$

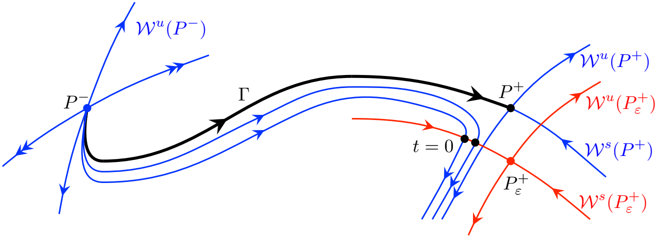

Figure 3.3: For $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+))$ there exists a continuous family of local defect solutions near P_ε^+ in a planar system. That is, $\mathcal{W}^u(P^-)$ and $\mathcal{W}^s(P_\varepsilon^-)$ intersect in a line (indicated by the black dots) near P_ε^+. Note that we can parametrize time in such a fashion that $t = 0$ coincides with an particular intersection of $\mathcal{W}^u(P^-)$ and $\mathcal{W}^s(P_\varepsilon^-)$ creating a local defect solution. In this figure, we have $\dim(\mathcal{W}^u(P^-)) = 2 > 1 = \dim(\mathcal{W}^u(P^+))$. See Figure 3.4 for sketches of the case $\dim(\mathcal{W}^u(P^-)) = 1 > 0 = \dim(\mathcal{W}^u(P^+))$.

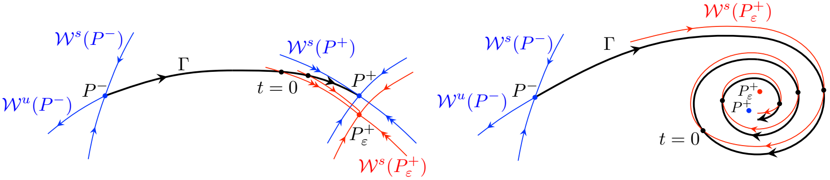

Figure 3.4: For $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+))$ there exists a continuous family of local defect solutions near P_ε^+ in a planar system. In this figure, we have $\dim(\mathcal{W}^u(P^-)) = 1 > 0 = \dim(\mathcal{W}^u(P^+))$. Left panel: P^+ has two real negative simple eigenvalues. Right panel: P^+ has a complex pair of eigenvalues with negative real part. See also Figure 3.3.

- $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+))$

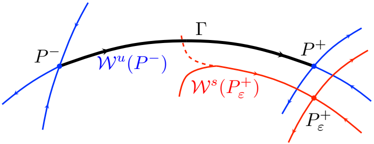

Figure 3.5: Left panel: for $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 1$, a generic perturbation does not lead to a local defect solution in a planar system near P_ε^+ since the stable and unstable manifold of P^+ and P_ε^+ are locally *parallel* and therefore Γ does not intersect with $\mathcal{W}^s(P_\varepsilon^+)$ near P_ε^+. A trivial defect solution connecting P^+ to P_ε^+ does exist since $\mathcal{W}^u(P^+)$ and $\mathcal{W}^s(P_\varepsilon^+)$ intersect for generic perturbations and also global defect solution can of course exist (this is indicated by the red dotted trajectory). Right panel: in the case of $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 1$ a non-generic perturbation can lead to non-generic local defect solution in a planar system.

We summarize the results for local defect solutions near P_ε^+ for (1.1) with $n = 2$ in Table 1.

Two dimensional systems with heteroclinic connection

$\dim(\mathcal{W}^u(P^-))$	$\dim(\mathcal{W}^u(P^+))$	
	0	1
1	cont. family	none
2	cont. family	cont. family

Table 1: Local defect solutions near P_ε^+ in generically perturbed two-dimensional systems.

3D Case.

- $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+))$

A continuous family of local defect solutions.
Since $\dim(\mathcal{W}^u(P^-)) + \dim(\mathcal{W}^s(P_\varepsilon^+)) \geq 4$, we have that the intersection $\mathcal{W}^u(P^-) \cap \mathcal{W}^s(P_\varepsilon^+)$ in a three-dimensional space is generically at least a one parameter family of solutions. See Theorem 4.1.

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Two one-dimensional curves in a three-dimensional space generically do not intersect and since $\mathcal{W}^s(P^+)$ and $\mathcal{W}^s(P_\varepsilon^+)$ are *locally parallel*, local (and global) defect solutions are not expected. See also Theorem 4.1.

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- $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+))$
  - $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 1$

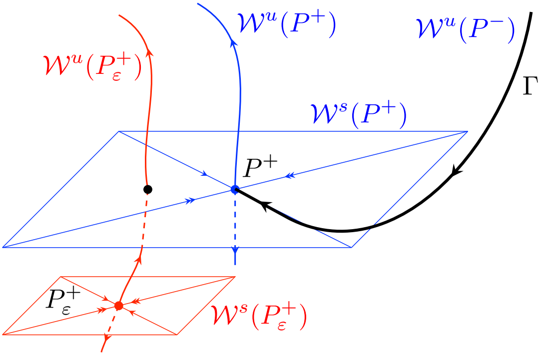

Figure 3.6: Also in three dimensional systems local defect solutions near P_ε^+ do generically not exist for $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 1$, since, generically, there exists no intersection of $\mathcal{W}^u(P^-)$ and $\mathcal{W}^s_{\text{loc}}(P_\varepsilon^+)$. The figure sketches the situation in the case that the stable eigenvalues of P^+ are real and simple.

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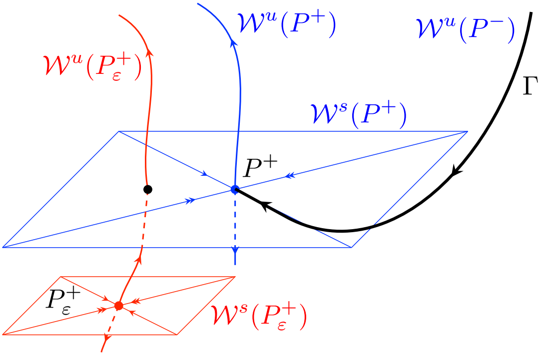

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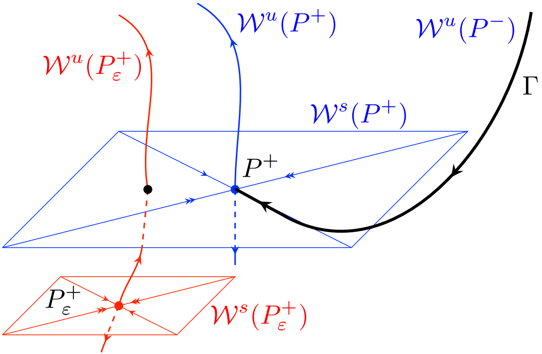

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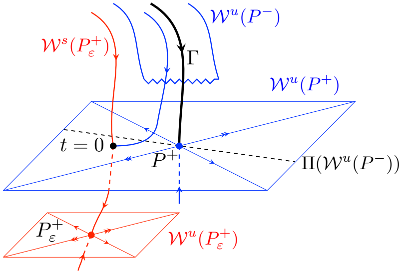
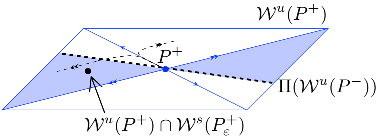
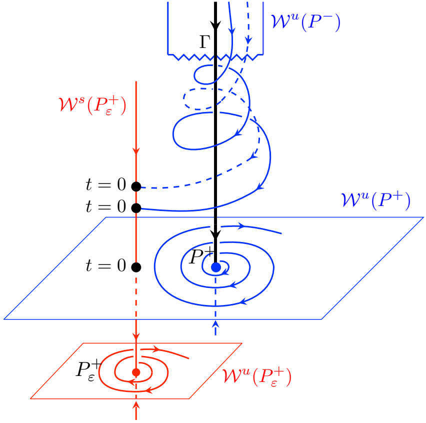
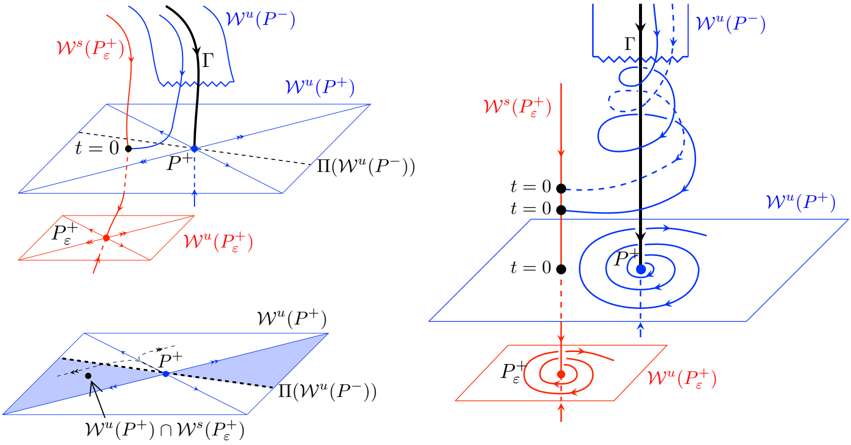
- $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 2$







- $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 2$



We summarize the results for local defect solutions near P_ε^+ for (1.1) with $n = 3$ in Table 2.

Three dimensional systems with heteroclinic connection

$\dim(\mathcal{W}^u(P^-))$	$\dim(\mathcal{W}^u(P^+))$		
	0	1	2
1	cont. family	none	none
2	cont. family	cont. family	real evals.: semi-cone condition complex evals.: countably many
3	cont. family	cont. family	cont. family

Table 2: Local defect solutions near P_ε^+ in generically perturbed three-dimensional systems.

OVERVIEW

Lemma 4.1

Assume Hypothesis (H1) hold and that P^+ *is a hyperbolic equilibrium point of (2.1). Then, there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *the system* (1.1) *has a unique trivial defect solution* $\Gamma_\varepsilon(t)$ *connecting* P^+ *and* P_ε^+, *where* $P_0^+ = P^+$.

back

DIMENSIONAL DEPENDENCE

Theorem 4.1

Assume Hypotheses (H1)-(H3) hold. Then, for $\varepsilon > 0$ small enough a necessary condition for having local defect solutions near P_ε^+ connecting P^- to P_ε^+ in (1.1) is

$$\dim(\mathcal{W}^u(P^-)) \geq \dim(\mathcal{W}^u(P^+)).$$

Moreover, if $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+))$ then we necessarily also need $\dim(\mathcal{W}^u(P^-)) > 1$ for a local defect solutions near P_ε^+ to exist. Finally, if $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+))$, then the necessary condition is also sufficient and (1.1) possesses a continuous family of local defect solutions near P_ε^+.

LOCAL WEAK DEFECTS

Theorem 4.2

Assume Hypotheses (H1)-(H3) hold and that

$$\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) > 1.$$

Then, for $\varepsilon > 0$ small enough and if the leading unstable eigenvalue of the Jacobian of P^+ is complex-valued, then (2.1) possesses countably many local defect solutions near P_ε^+. If the leading unstable eigenvalue of the Jacobian of P^+ is real-valued, then (2.1) possesses local defect solutions near P_ε^+ as long as the perturbation meets (a) semi-cone condition(s).

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Normal Form

Lemma 4.2

(Homburg and Sandstede 2010) *There exists a smooth coordinate transformation* $u \mapsto (x^{ls}, x^{ss}, y^{lu}, y^{uu})$ *such that (2.1) near the origin transforms into*

$$\begin{cases} \dot{x}^{ls} = A^{ls}x^{ls} + \mathcal{O}\left((|x^{ls}|^2 + |x^{ss}|)|y|\right), \\ \dot{x}^{ss} = A^{ss}x^{ss} + \mathcal{O}\left(|x^{ls}|^2 + |x^{ss}|(|x| + |y|)\right), \\ \dot{y}^{lu} = A^{lu}y^{lu} + \mathcal{O}\left((|x^{lu}|^2 + |y^{uu}|)|x|\right), \\ \dot{y}^{uu} = A^{uu}y^{uu} + \mathcal{O}\left(|x^{lu}|^2 + |y^{uu}|(|x| + |y|)\right), \end{cases} \tag{4.1}$$

where $x = (x^{ls}, x^{ss}) \in E^{ls} \oplus E^{ss}$ *and* $y = (y^{lu}, y^{uu}) \in E^{lu} \oplus E^{uu}$.

back

CASE I: THE UNIQUE LEADING UNSTABLE EIGENVALUE IS REAL AND SIMPLE

Let the section $L_{\vec{\delta}}$ be the intersection of $x = \vec{\delta}(\varepsilon)$ and $\mathcal{W}^u(P^-)$. In the δ-neighborhood of the origin, $L_{\vec{\delta}}$ can be expressed as

$$K_1 y^{lu} + \langle K_2, y^{uu} \rangle + \mathcal{O}(|y|^2) = 0, \tag{4.2}$$

where K_1 is constant, K_2 is a $(n - \ell - 1)$-dimension vector.

Theorem 4.3

Assume Hypotheses (H1)-(H3) hold and that the Jacobian of P^+ of system (2.1) has a real and simple leading unstable eigenvalue. Then, for sufficiently small $\varepsilon > 0$, (1.1) has at least a local defect solution near $P^+_\varepsilon = (x_*, y_*)$ if the following semi-cone condition is met

$$K_1 y_*^{lu} \left(K_1 y_*^{lu} + \langle K_2, y_*^{uu} \rangle \right) < 0. \tag{4.3}$$

Case I: The unique leading unstable eigenvalue is real and simple

Theorem 4.4

Assume Hypotheses (H1)-(H3) hold and that the Jacobian of the P^+ of system (1.1) has m distinct real unstable eigenvalues. Let $P_\varepsilon^+ = (x_*(\varepsilon), y_*(\varepsilon))$ be the equilibrium of (2.2). Then, for sufficiently small $\varepsilon > 0$ there exist regions Ω_k in the space of parameters x_*, y_*, K_1 and K_2 such that system (1.1) has k local defect solutions connecting P^- to P_ε^+, where $k = 0, 1, \cdots, m-1$.

Case I: The unique leading unstable eigenvalue is real and simple

Theorem 4.5

Assume Hypotheses (H1)-(H3) hold and that the Jacobian of P^+ of system (2.1) has m distinct unstable eigenvalues, among which there is a real leading unstable eigenvalue and at least a pair of complex conjugate unstable eigenvalues. Let $P_\varepsilon^+ = (x_*(\varepsilon), y_*(\varepsilon))$ be the equilibrium of (2.2). Then, for any $k \in \mathbb{Z}^+$ there exists a region Ω_k for x_*, y_*, K_1 and K_2 such that for sufficiently small $\varepsilon > 0$ system (1.1) has k local defect solutions connecting P^- to P_ε^+.

CASE II: THE LEADING UNSTABLE EIGENVALUES ARE A PAIR OF COMPLEX CONJUGATION

Fix δ small enough and let the section $L_{\vec{\delta}}$ be the intersection of $x = \vec{\delta}(\varepsilon)$ with $|\vec{\delta}| = \delta$ and $\mathcal{W}^u(P^-)$. In the δ-neighborhood of the origin, $L_{\vec{\delta}}$ can be expressed as

$$\langle \overline{K}_1, y^{lu} \rangle + \langle \overline{K}_2, y^{uu} \rangle + \mathcal{O}(|y|^2) = 0, \tag{4.4}$$

where $\overline{K}_1$ and $\overline{K}_2$ are 2 and $(m-2)$ dimensional vectors, respectively.

Theorem 4.6

Assume Hypotheses (H1)-(H3) hold and $\left|\overline{K}_1\right| \neq 0$ *in (38). Let the Jacobian of* P^+ *of system (2.1) have a pair of complex conjugation and simple leading unstable eigenvalues. Then, for sufficiently small* $\varepsilon > 0$ *system (1.1) has countably infinite local defects near* P^+_ε*.*

CONTENTS

- Extended Fisher-Kolmogorov equation
- Perturbed FitzHugh-Nagumo equation

EXTENDED FISHER-KOLMOGOROV EQUATION

Consider the extended Fisher-Kolmogorov equation

$$\frac{\partial u}{\partial t} = -\hbar \frac{\partial^4 u}{\partial \xi^4} + \frac{\partial^2 u}{\partial \xi^2} + u - u^3, \quad \hbar > 0,$$

which was proposed as a higher order model equation for non-trivial spatio-temporal pattern formation by Dee and van Saarloos. Its stationary equation is

$$-\hbar \frac{d^4 u}{d\xi^4} + \frac{d^2 u}{d\xi^2} + u - u^3 = 0,$$

which by the change $\xi \to \hbar^{1/4}\xi$ can be transformed into the canonical form

$$\frac{d^4 u}{d\xi^4} + \beta \frac{d^2 u}{d\xi^2} + u - u^3 = 0, \quad \beta = -1/\sqrt{\hbar} < 0. \tag{5.1}$$

EXTENDED FISHER-KOLMOGOROV EQUATION

Consider an inhomogeneous perturbation of equation (5.1)

$$\frac{d^4u}{d\xi^4} + \beta\frac{d^2u}{d\xi^2} + u - u^3 = \begin{cases} 0, & \xi < 0, \\ \varepsilon g(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}), & \xi > 0. \end{cases} \tag{5.2}$$

The equivalent system of first order ODEs is given by

$$\begin{pmatrix} u' \\ p' \\ q' \\ r' \end{pmatrix} = \begin{cases} \begin{pmatrix} p \\ q \\ r \\ u - u^3 - \beta q \end{pmatrix}, & \xi < 0 \\ \begin{pmatrix} p \\ q \\ r \\ u - u^3 - \beta q + \varepsilon g(u, p, q, r). \end{pmatrix}, & \xi > 0. \end{cases} \tag{5.3}$$

Extended Fisher-Kolmogorov equation

(5.3)$_0$ has three equilibria $P^0 = (0, 0, 0, 0)$ and $P^{\pm} = (\pm 1, 0, 0, 0)$.

Corollary 5.1

For $\beta < 0$ and sufficiently small $\varepsilon > 0$ there exists a unique trivial defect solution in (5.3) that connects P^- to P^-_ε and a unique trivial defect solution that connects P^+ to P^+_ε.

Theorem 5.1

Let $\beta \in (-2\sqrt{2}, 0)$ in (5.3)$_0$. Then, there is an isolated heteroclinic solution $\Gamma_1 = (u_1, p_1, q_1, r_1)$ that connects P^- to P^+; the $u-$component $u_1(\xi - \xi_*)$ of Γ_1 corresponds to a translational family of kink solutions of (5.1) that have a unique zero at the midpoint $\xi = \xi_*$ and that are odd as a function of ξ w.r.t $\xi = \xi_*$.

[cf. L.A. Peletier and W.C. Troy, Spatial Patterns: Higher Order Models in Phys. and Mech., Birkhäuser, 2001.]

EXTENDED FISHER-KOLMOGOROV EQUATION

The equilibrium points $P^{\pm}$ persist in heterogeneously perturbed system (5.3)

$$P_{\varepsilon}^{\pm} = (\pm 1 + \varepsilon g(\pm 1, 0, 0, 0) + \mathcal{O}(\varepsilon^2), 0, 0, 0).$$

Theorem 5.2

Let $g(1,0,0,0) \neq 0$ and $\beta \in (-2\sqrt{2}, 0)$. Then, for $\varepsilon > 0$ small enough, the stationary perturbed heterogeneous eFK system (5.3) supports countably many local defect kink solutions that connect P^- to P_{ε}^{+}.

PERTURBED FITZHUGH-NAGUMO EQUATION

The perturbed generalized three-component FitzHugh-Nagumo system

$$\begin{aligned} U_t &= \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma(x)) \\ \tau V_t &= V_{xx} + U - V \\ \theta W_t &= D^2 W_{xx} + U - W \,. \end{aligned} \tag{5.4}$$

Written as a system of six first order ODEs, it is given by

$$\begin{cases} u_\xi &= p\,, \\ p_\xi &= -u + u^3 + \varepsilon(\alpha v + \beta w + \gamma(\xi))\,, \\ v_\xi &= \varepsilon q\,, \\ q_\xi &= \varepsilon(v - u)\,, \\ w_\xi &= \frac{\varepsilon}{D} r\,, \\ r_\xi &= \frac{\varepsilon}{D}(w - u)\,, \end{cases} \tag{5.5}$$

with $\xi := x/\varepsilon, \alpha, \beta \in \mathbb{R}, D > 1, 0 < \varepsilon \ll 1$, and $\gamma(\xi)$ as in (1.3).

PERTURBED FITZHUGH-NAGUMO EQUATION

The perturbed generalized three-component FitzHugh-Nagumo system

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with $\xi := x/\varepsilon, \alpha, \beta \in \mathbb{R}, D > 1, 0 < \varepsilon \ll 1$, and $\gamma(\xi)$ as in (1.3).

PERTURBED FITZHUGH-NAGUMO EQUATION

Its reduced two-component model

$$\left\{\begin{array}{rcrcl} U_t & = & \varepsilon^2 U_{xx} & + & U - U^3 - \varepsilon(\alpha V + \gamma(x)), \\ \tau V_t & = & V_{xx} & + & U - V, \end{array}\right. \tag{5.6}$$

with $(x,t) \in (\mathbb{R}, \mathbb{R}^+), \alpha \in \mathbb{R}, D > 1, \tau > 0, 0 < \varepsilon \ll 1$, and $\gamma(x)$ as in (1.3).

The associated ODE is four dimensional.

$$\left\{\begin{array}{rcl} u_\xi & = & p\,, \\ p_\xi & = & -u + u^3 + \varepsilon(\alpha v + \gamma(\xi))\,, \\ v_\xi & = & \varepsilon q\,, \\ q_\xi & = & \varepsilon(v - u)\,. \end{array}\right. \tag{5.7}$$

Theorem 5.3

Let $\gamma(\xi)$ be as in (1.3) with $\gamma_1 = 0$ and let ε be small enough. Moreover, let $\Gamma_{\text{het}}(\xi)$ be the 1-front heteroclinic orbit that connects $P_1(0) \equiv P^-$ to $P_2(0) \equiv P^+$ in the homogeneous case $\gamma = 0$, and let $\alpha, \beta, \gamma_2 \in \mathbb{R}$ be $\mathcal{O}(1)$ with respect to ε.

- *$\ell = 2$: Then, there exists a local defect heteroclinic orbit $\Gamma_{\text{het,defect}}(\xi)$ to (5.7) that connects P^- to $\tilde{P}^+$ if and only if $\alpha > 0$;*
- *$\ell = 3$: Then, there exists a local defect heteroclinic orbit $\Gamma_{\text{het,defect}}(\xi)$ to (5.5) that connects P^- to $\tilde{P}^+$ for $\alpha, \beta > 0$.*

The orbit of the local defect $\Gamma_{\text{het,defect}}(\xi)$ in the 2ℓ-dimensional phase space of (5.7)/(5.5) is $O(\varepsilon)$-close to the corresponding $\Gamma_{\text{het}}(\xi)$.

PERTURBED FITZHUGH-NAGUMO EQUATION

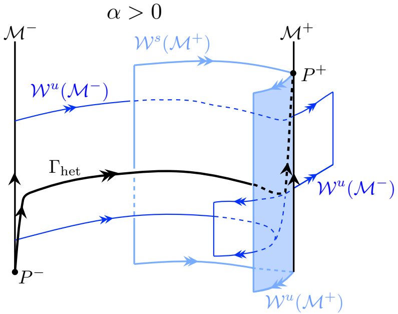

Figure 5.1: The three dimensional unstable manifold $\mathcal{W}^u(\mathcal{M}^-)$ and stable manifold $\mathcal{W}^s(\mathcal{M}^+)$ of the two dimensional slow manifolds $\mathcal{M}^-$ and $\mathcal{M}^+$ in the four dimensional phase space associated to (5.7), sketched as two dimensional unstable and stable manifolds in $\mathbb{R}^3$. Left panel: $\alpha > 0$ and $\mathcal{W}^u(\mathcal{M}^-)$ is outside $\mathcal{W}^u(\mathcal{M}^+) \cup \mathcal{W}^s(\mathcal{M}^+)$ for $v_0 > 0$ and inside for $v_0 < 0$. Right panel: $\alpha < 0$ and note that the more subtle stretched and folded structure of $\mathcal{W}^u(\mathcal{M}^-)$ exponentially close to $\mathcal{W}^u(\mathcal{M}^-) \cap \mathcal{W}^s(\mathcal{M}^+)$ is not shown.

Thank you!