On periodic orbits in non-smooth and singularly perturbed differential equations with applications

R. Prohens

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Prof. Jiaqi Mo's 80th birthday (Happy birthday)

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On periodic orbits in non-smooth singularly perturbed differential equations with applications

Main purposes of this talk:

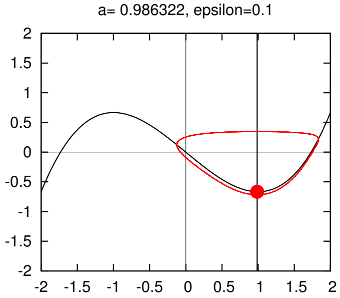
- Reproduce smooth slow-fast dynamics in the framework of PWL
- Approachable study of periodic behaviours, MMOs, through PWL slow-fast differential systems
- Present simplified models, meaningful for neuroscience applications

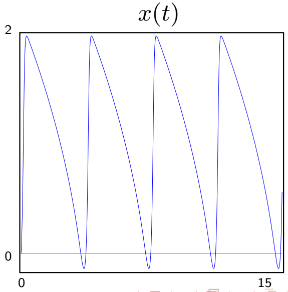
1. Introduction
   - Canard cycles. References
2. Qualitative
   - Maximal and faux canards in $\mathbb{R}^n$
   - MMOs in $\mathbb{R}^3$
3. Quantitative
   - Synaptic conductances estimation in a McKean neuron model

Introduction: Canard cycles

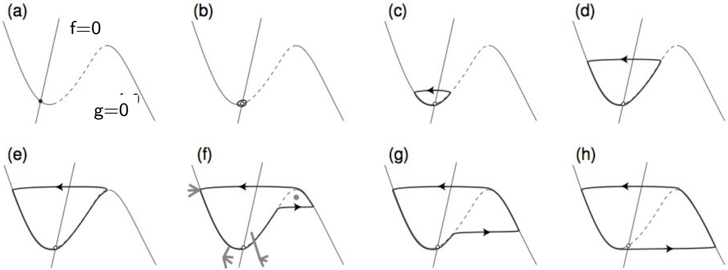
- Canard cycles: Limit cycles flowing with very different velocities along the orbit.
- Tipically occur in slow-fast differential systems.
- Exemple: Van der Pol system

$$\left.\begin{array}{l}\dot{x} = y - f(x)\\ \dot{y} = \varepsilon(a - x)\end{array}\right\}, \quad f(x) = x\left(\frac{x^2}{3} - 1\right), \quad 0 < \varepsilon \ll 1$$





Canard Orbits in 2-D



[Izhikevich, Springer(2007)]

Canard orbits cross from the attracting manifold to the repelling manifold.

The first existence results on canards were obtained by means of nonstandard analytical techniques in Benoît et al.[1]

Canards also where reported and studied by Dumortier and Roussarie, by using invariant manifold theory and parameter blow-up[2]

[1] Benoît E, Callot J-L, Diener F, Diener M. 1981 *Chasse au canard*. Coll. Maths 32, 37–119.

[2] Canard Cycles and Center Manifolds. **577**, Memoirs of the American Mathematical Society, (1996).

Introduction: Canard explosion

- Canard explosion: Sudden growing of the amplitude of the limit cycle.
- Exemple: Van der Pol system

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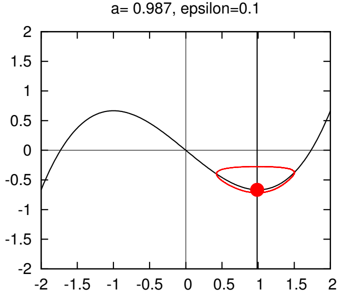

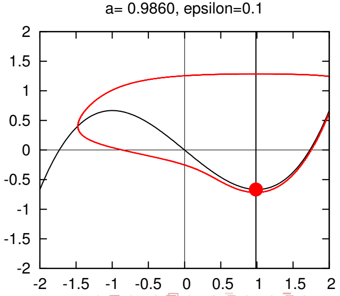

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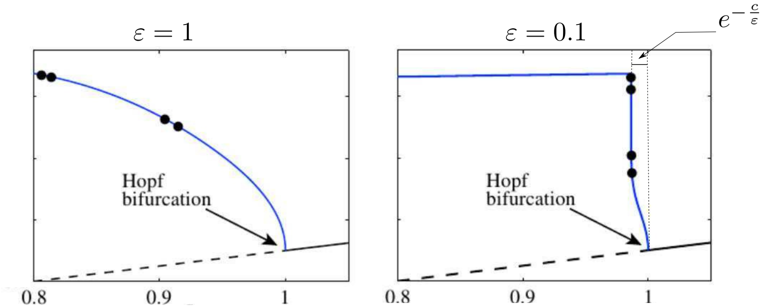
$$\left.\begin{array}{l}\dot{x} = y - f(x)\\ \dot{y} = \varepsilon(a - x)\end{array}\right\}, \quad f(x) = x\left(\frac{x^2}{3} - 1\right), \quad J = \begin{pmatrix} 1 - a^2 & 1 \\ -\varepsilon & 0 \end{pmatrix}$$

$$\triangle = (1 - a^2)^2 - 4\varepsilon$$



Fenichel's geometric theory allows us to analyze the dynamics of

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \varepsilon g(\mathbf{u}, \mathbf{v}, \varepsilon), \quad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = f(\mathbf{u}, \mathbf{v}, \varepsilon), \qquad \text{(fast time scale)}$$

where $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^s \times \mathbb{R}^q$ when f and g are sufficiently smooth functions. The coordinates of $\mathbf{u}$ are called *slow variables*, while the coordinates of $\mathbf{v}$ are called *fast variables*.

Or equivalently, $\tau = \varepsilon t$,

$$\mathbf{u}' = \frac{d\mathbf{u}}{d\tau} = g(\mathbf{u}, \mathbf{v}, \varepsilon), \quad \varepsilon\mathbf{v}' = \varepsilon\frac{d\mathbf{v}}{d\tau} = f(\mathbf{u}, \mathbf{v}, \varepsilon). \qquad \text{(slow time scale)}$$

This analysis follows by combining the behaviour of the singular orbits, corresponding to the limiting problems, $\varepsilon = 0$

$$\begin{array}{lll} \text{layer:} & \dot{\mathbf{u}} = \mathbf{0}, & \dot{\mathbf{v}} = f(\mathbf{u}, \mathbf{v}, 0), \\ \text{reduced:} & \mathbf{u}' = g(\mathbf{u}, \mathbf{v}, 0), & \mathbf{0} = f(\mathbf{u}, \mathbf{v}, 0), \quad \mathbf{u} \in \mathbb{R}^s \end{array}$$

where $' = d/d\tau$, $\tau = \varepsilon t$. Critical manifold

$$\mathcal{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{s+q} \,|\, f(\mathbf{u}, \mathbf{v}, 0) = \mathbf{0}\}.$$

We call normally hyperbolic the singular points $(\mathbf{u}_0, \mathbf{v}_0) \in \mathcal{S}$ for which the eigenvalues of the Jacobian matrix $D_{\mathbf{v}} f(\mathbf{u}_0, \mathbf{v}_0)$ have nonzero real part.

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Question

What does it remains in the dynamical behaviour when smoothness is no longer present?

That is, what does it remains from previous dynamic behavior when smoothness on the critical manifold $\mathcal{S}$ is not assumed?

$$\mathcal{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{s+q} \,|\, f(\mathbf{u}, \mathbf{v}, 0) = \mathbf{0}\}.$$

For instance, for $s = 1$ and $q = 1$...

Related works

Canard dynamics has been investigated in planar PWL slow-fast systems, from the 1990s.

Canard-type behaviour has been observed experimental electronic circuits

The corners play the role of the fold points and cycles resembling canards and evolving around these corners were identified by simulation.

1991 M. Itoh and R. Tomiyasu, *Canards and irregular oscillations in a nonlinear circuit*, in Circuits and Systems, 1991., IEEE International Sympoisum on, IEEE, 1991, pp. 850–853.

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The cubic critical manifold is replaced by a PWL caricature consisting of three straight line segments.

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Canards with head can arise only in systems with one more piece in between the two corners.

The main idea to obtain true canard cycles in a planar PWL systems consists in approximating the critical manifold near a fold by a three-piece PWL function.

1997 N. Arima, H. Okazaki, H. Nakano, *A generation mechanism of canards in a piecewise linear system*, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences 80 (1997) 447–453.

To obtain the correct transition of eigenvalues near the fold.

In [PRSZ2011][3] the authors prove the existence of canard cycles in singularly perturbed piecewise differential systems with $s = 2$ and $q = 1$.

This fact suggested that: canards are not exclusively a differential phenomenon, but rather a geometric one.

[3] A. Pokrovskii, D. Rachinskii, V. Sobolev and A. Zhezherun, *Topological degree in analysis of canard-type trajectories in 3-D systems*, Applicable Analysis: An International Journal, 90 (2011), 1123–1139.

More recently:

- 2011 D. J. Simpson and R. Kuske, *Mixed-mode oscillations in a stochastic, piecewise-linear system*, Physica D, 240 (2011), pp. 1189–1198.
- 2012 H. G. Rotstein, S. Coombes, and A. M. Gheorge, *Canard-like explosion of limit cycles in two-dimensional piecewise-linear models of FitzHugh-Nagumo type*, SIAM Journal on Applied Dynamical Systems, 11 (2012), pp. 135–180.
- 2013 M. Desroches, E. Freire, S. J. Hogan, E. Ponce, P. Thota, *Canards in piecewise-linear systems: explosions and super-explosions*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 469 (2013).

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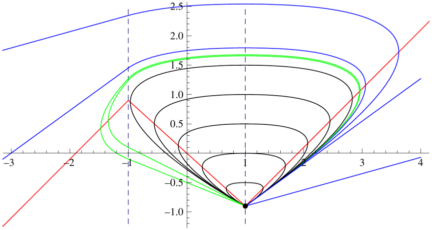
Planar slow-fast piecewise-linear (PWL) system with three zones admits limit cycles that share a lot of similarity with van der Pol canards, in particular and explosive growth.

$$\left.\begin{array}{l} \dot{x} = y - f(x) \\ \dot{y} = \varepsilon(a - x) \end{array}\right\}$$

$$f(x) = \begin{cases} x + k + 1 & x < -1, \\ -kx & |x| < 1, \\ -x - k - 1 & x > 1 \end{cases}$$

$$\triangle = k^2 - 4\varepsilon > 0$$

$a = 1$



- M. Desroches, E. Freire, S.J. Hogan, E. Ponce and P. Thota *Canards in PWL systems: explosions and super-explosions*, Proc. Royal S. 2013.

Absence of true canards with still the presence of an explosive growth of cycles upon parameter variation.

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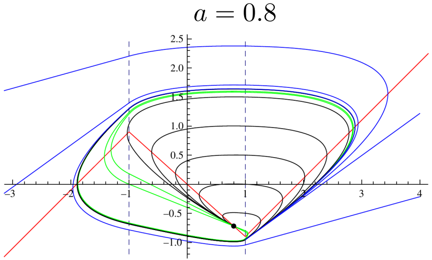

References on neuron models

A large class of neuron models are based on the approximation that the membrane of the neuron behaves like a circuit. The voltage equation is obtained by applying Kirchoff's law.

After the model by (HH model):

1952 A. L. Hodgkin and A. F. Huxley, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, The Journal of physiology, 117 (1952), p. 500.

The first reduction to a planar system (FHN model):

1961 R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical Journal, 1 (1961), pp. 445–466.

1962 J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proceedings of the IRE, 50 (1962), pp. 206–2070.

where, the vector field of the HH model was approximated by a polynomial system through the crucial observation that the voltage nullcline is roughly cubic shaped.

Hence, the FHN model appears as a modified van der Pol system.

The FHN model was investigated from the slow-fast perspective and further simplified by approximating the cubic voltage nullcline by a PWL function in

1970 M. P. McKean, Nagumo's equation, Advances in mathematics, 4 (1970), pp. 209–223, The McKean model, is a two dimensional FHN model of speaking neurons

$$C\frac{dv}{dt} = f(v) - w + I(t), \quad \frac{dw}{dt} = v - \gamma w,$$

where $I(t)$ is the external current and $f(v)$ is a piecewise linear caricature of the cubic FitzHugh-Nagumo function

$$f(v) = \begin{cases} -v & < a/2, \\ v - a & a/2 \leq v \leq (1+a)/2, \\ 1 - v & v > (1+a)/2. \end{cases}$$

$v(t)$ is the membrane potential, $w(t)$ is the membrane current.

a, $\gamma > 0$ constants

C corresponds to the cell membrane capacitance which is assumed to be small and bounded $0 < C \ll 0.1$.

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$$\mathcal{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{s+q} \,|\, f(\mathbf{u}, \mathbf{v}, 0) = \mathbf{0}\}.$$

For instance, for $s \geq 2$, $q = 1$.

- R.P., A.E.Teruel
  Canard trajectories in 3D piecewise linear systems,
  Discrete and Continuous Dynamical Systems. 2013

For next singularly perturbed 3–dimensional piecewise linear differential system

$$\begin{cases} \dot{u}_1 = \varepsilon(a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1), \\ \dot{u}_2 = \varepsilon(a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2), \\ \dot{v} = u_1 + |v|, \end{cases}$$

where $0 < \varepsilon \ll 1$,

- we provide numerical arguments for the existence of a canard cycle.

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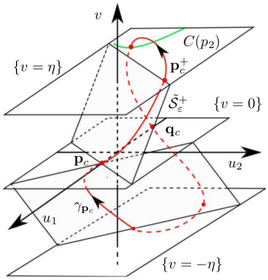

Representation of the canard cycle $\gamma_{\mathbf{p}_c}$, slow manifolds $\tilde{\mathcal{S}}_\varepsilon^- \cup \tilde{\mathcal{S}}_\varepsilon^+$ and the border planes $\{v=\eta\}, \{v=0\}$ and $\{v=-\eta\}$, which separate the regions where the system is linear. We highlighted the points of intersection of $\gamma_{\mathbf{p}_c}$ with the border planes.

Qualitative compatibility between PWL and smooth diff. eq.

Since the late 1990s several papers have shown that the canard phenomenon can be reproduced with piecewise-linear (PWL) dynamical systems in two and three dimensions, exhibiting an slow-fast dynamics.

- R.P., A.E.Teruel, C. Vich
  Slow-fast n-dimensional piecewise linear differential systems,
  Jour. Diff. Eq. 2016.

Slow-Fast Piecewise Linear System (PWLS)

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \varepsilon(A\mathbf{u} + \mathbf{a}v + \mathbf{b}), \qquad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = u_1 + |v|.$$

This paper is mainly concerned with maximal canard orbits occurring in n-dimensional piecewise linear slow-fast systems.

More precisely, conditions for the existence of maximal canard orbits and/or faux canard orbits are established.

We show that these maximal canards perturb from singular orbits (singular canards) whose order of contact with the fold manifold is greater than or equal to two.

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$\mathbf{u} \in \mathbb{R}^s$ slow variable
$\mathbf{v} \in \mathbb{R}$ fast variable

$0 < \varepsilon \ll 1$ ratio of time scales
$n = s + 1$ system dimension

$A = (a_{ij})_{1 \le i,j \le s}$ $s \times s$ real matrix
$\mathbf{a} = (a_1, a_2, \ldots, a_s)^T$ vector in $\mathbb{R}^s$
$\mathbf{b} = (b_1, b_2, \ldots, b_s)^T$ vector in $\mathbb{R}^s$

- Rather general: $f(\mathbf{u}, v, \varepsilon) = \mathbf{d}^T\mathbf{u} + |v|$, with $\mathbf{d} \neq \mathbf{0}$, can be transformed into our system $(\mathbf{u} \to (\mathbf{d}^T\mathbf{u}, u_2, \ldots, u_n)^T)$.

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- 2 regimes: $\{v \le 0\}$ and $\{v \ge 0\}$ and 1 common boundary $\{v = 0\}$.

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- Slow-Fast PWLS dynamics approach:

Associated we have the:

- fast subsystem (layer problem)
- slow subsystem (reduced problem)
- critical manifold, where the slow subsystem is defined, $\mathcal{S}$
- fold manifold, $\mathcal{F}$, when normal hyperbolicity fails, (points where $\mathcal{S}$ folds)

Unperturbed Dynamics:

- Fast subsystem

$$\begin{cases} \dot{\mathbf{u}} = \mathbf{0}, \\ \dot{v} = u_1 + |v|, \end{cases}$$

- Critical manifold

$$\mathcal{S} = \{(\mathbf{u}, v) \in \mathbb{R}^n : \; u_1 + |v| = 0\}$$

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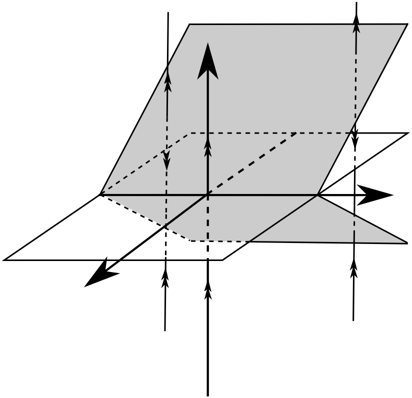
- $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{F} \cup \mathcal{S}^-$

$$\mathcal{S}^+ = \{u_1 + v = 0; \ v > 0\}$$
$$\mathcal{S}^- = \{u_1 - v = 0; \ v < 0\}$$
$$\mathcal{F} = \{u_1 = 0, \ v = 0\}$$

where $\mathcal{S}^+$ and $\mathcal{S}^-$ are normally hyperbolic and $\mathcal{F}$ is the fold manifold



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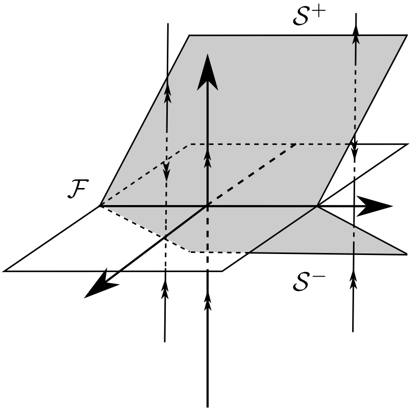


Unperturbed Dynamics:

- Slow subsystem associated to

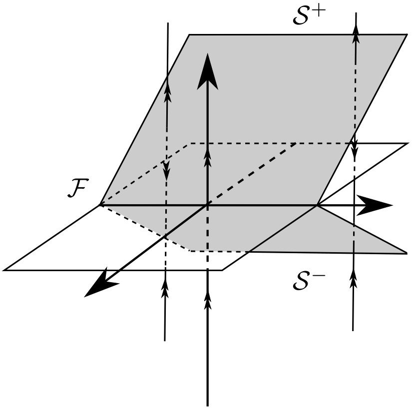
$$\begin{cases} \mathbf{u}' = A\mathbf{u} + \mathbf{a}v + \mathbf{b}, \\ \varepsilon v' = u_1 + |v|, \end{cases}$$

Unperturbed Dynamics:

- Slow subsystem

$$\begin{cases} \mathbf{u}' = A\mathbf{u} + \mathbf{a}v + \mathbf{b}, \\ 0 = u_1 + |v|, \end{cases}$$

The slow subsystem is a linear differential equation defined on the critical manifold $\mathcal{S}$, but it is not defined on $\mathcal{F}$. To overcome this problem, we consider the Filippov's convention.



Unperturbed systems

- Slow subsystem assocciated to

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Unperturbed systems

- Slow subsystem

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To analyze its dynamics, let us consider a locally conjugate system defined on $\mathbb{R}^s \setminus \{\mathbf{e}_1^T \tilde{\mathbf{u}} = 0\}$

Unperturbed systems

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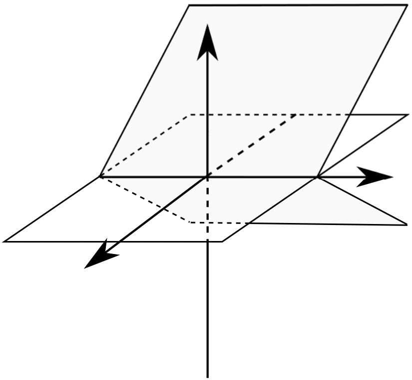
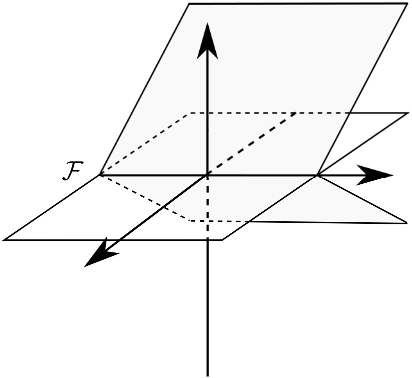
To analyze its dynamics, let us consider a locally conjugate system defined on $\mathbb{R}^s \setminus \{\mathbf{e}_1^T \tilde{\mathbf{u}} = 0\}$



Unperturbed systems

- Slow subsystem in $\mathcal{S} \setminus \mathcal{F}$

$$\begin{cases} \mathbf{u}' = A\mathbf{u} + \mathbf{a}v + \mathbf{b}, \\ v' = \dfrac{|v|}{v} u_1', \end{cases}$$



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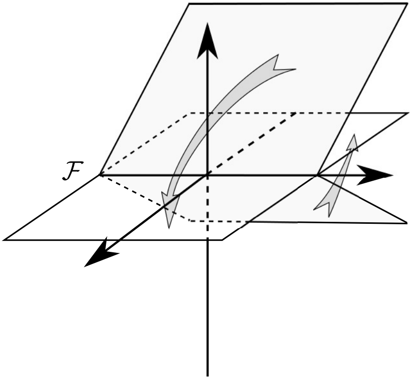
$$\tilde{\mathbf{u}} = \pi(\mathbf{u}, v) = \begin{cases} P\mathbf{u} & \text{if } v > 0, \\ \mathbf{u} & \text{if } v < 0 \end{cases}$$

where $P = I - 2\mathbf{e}_1\mathbf{e}_1^T$.



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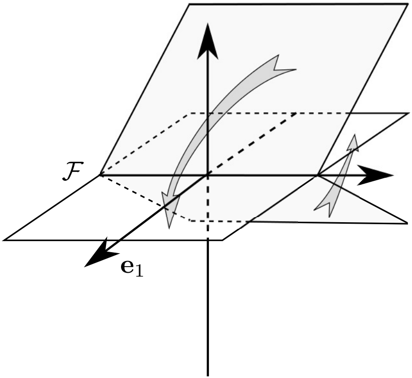

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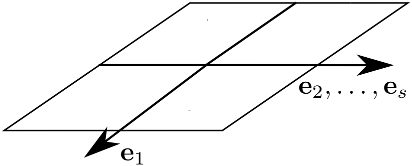

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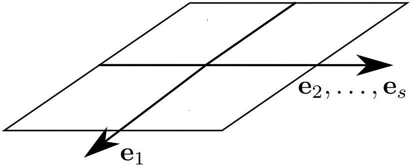

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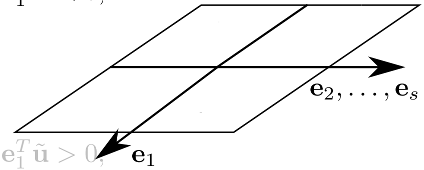

- By adopting Filippov's convention

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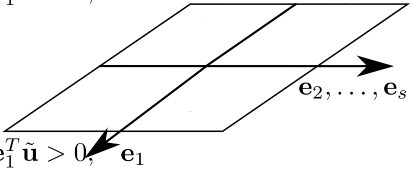

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Singularly Perturbed System

Slow-Fast Piecewise Linear System (PWLS)

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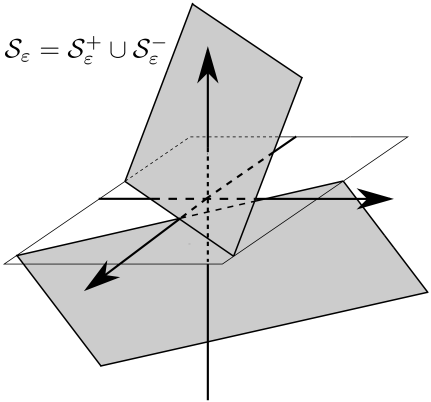

- The manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$ is a Fenichel's manifold.
- We get the expression of $\mathcal{S}_\varepsilon$.

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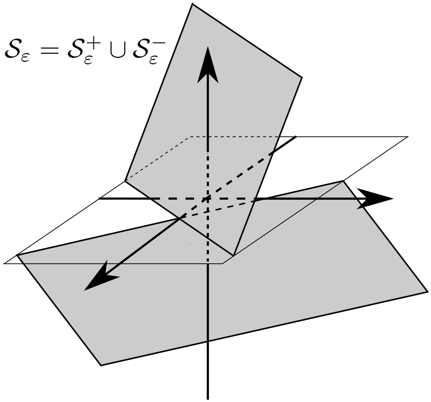

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Singularly Perturbed System

Theorem 1.

The manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$ is a Fenichel's manifold.

- $$\mathcal{S}_\varepsilon^+ = \Big\{(\mathbf{u}, v) \in \mathbb{R}^n : \; v \geq 0, \\ -\mathbf{e}_1^T(\varepsilon A - \lambda_n^+ I)^{-1}\mathbf{u} + v = \frac{\varepsilon}{\lambda_n^+}\mathbf{e}_1^T(\varepsilon A - \lambda_n^+ I)^{-1}\mathbf{b}\Big\}.$$
- $$\mathcal{S}_\varepsilon^- = \Big\{(\mathbf{u}, v) \in \mathbb{R}^n : \; v \leq 0, \\ -\mathbf{e}_1^T(\varepsilon A - \lambda_n^- I)^{-1}\mathbf{u} + v = \frac{\varepsilon}{\lambda_n^-}\mathbf{e}_1^T(\varepsilon A - \lambda_n^- I)^{-1}\mathbf{b}\Big\}.$$

For $\varepsilon > 0$ and sufficiently small, $\mathcal{S}_\varepsilon$ satisfies:

a) $\mathcal{S}_\varepsilon$ is locally invariant manifold.

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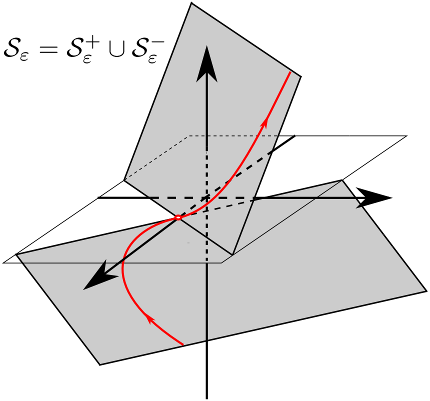
Maximal Canard Orbits

A point $\mathbf{p}_\varepsilon$ in $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$, it is said to be a maximal canard (resp. faux canard) point if the orbit, $\gamma_{\mathbf{p}_\varepsilon}$, through $\mathbf{p}_\varepsilon$ is a maximal canard (resp. faux canard) orbit.



- Maximal canard orbits cross from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^+$.
- To locate them we study:
  - Behaviour of the flow on $\mathcal{F}$,
  - order of contact of $\mathbf{p}_\varepsilon \in \mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$.

Proposition. Order of contact

k is the order of contact of the flow with the switching manifold $\{v=0\}$ at $\mathbf{p}=(\mathbf{u},0)\in\mathcal{P}$ if

$k=1$

$$\mathbf{e}_n^T\left(\begin{pmatrix}\varepsilon A & \varepsilon\mathbf{a}\\ \mathbf{e}_1^T & \pm 1\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ 0\end{pmatrix}+\begin{pmatrix}\varepsilon\mathbf{b}\\ 0\end{pmatrix}\right)=\mathbf{e}_1^T\mathbf{u}\neq 0$$

$k\geq 2$

$$\mathbf{e}_n^T\left(B_\varepsilon^\pm\right)^r\left(B_\varepsilon^\pm\mathbf{p}+\mathbf{b}_\varepsilon\right)=0,\quad r=0,1,\ldots,k-2$$

$$s=\mathbf{e}_n^T\left(B_\varepsilon^\pm\right)^{k-1}\left(B_\varepsilon^\pm\mathbf{p}+\mathbf{b}_\varepsilon\right)\neq 0.$$

- At even contact points, the flow does not cross $\mathcal{P}$
- At odd contact points, the flow crosses $\mathcal{P}$
- Maximal canard orbits cross throught odd contact points with $s>0$.

Intersection point $\mathbf{p}_\varepsilon$

Study of $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$

Theorem 2. $s \geq 2$ and $a_{1j} \neq 0$ for some $j \neq 1$

a) $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$ is a $(n-3)$-D LM. $\mathbf{p}_\varepsilon = (\mathbf{u}, 0) \in \mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$

$$u_1 = \frac{\varepsilon^2}{\lambda_n^+ \lambda_n^-} \mathbf{e}_1^T A \left(A\tilde{\mathbf{u}} + \mathbf{b} \right)$$

$$u_j = -\frac{1}{a_{1j}} \left(b_1 + \sum_{k=2,\neq j}^{s} a_{1k} u_k \right) + O(\varepsilon)$$

Existence of Maximal Canard Orbits.

Study of $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$

Theorem 2

b) If $a_{1j} \neq 0$ for some $j \in \{2, \ldots, s\}$, then $\dim(\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-) = n - 3$
- b.1) If $u_1 > 0$, $\exists$ maximal canard through $\mathbf{p}_\varepsilon$ and order of contact 1;
- b.2) If $u_1 < 0$, $\exists$ faux canard through $\mathbf{p}_\varepsilon$ and order of contact 1;
- b.3) If $u_1 = 0$, order of contact greater than or equal to 2.

c) If $a_{1j} = 0$ for all $j \in \{2, \ldots, s\}$ and $b_1 = 0$, then $\dim(\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-) = n - 2$ and neither maximal nor faux canard orbits exist. Invariant manifold.

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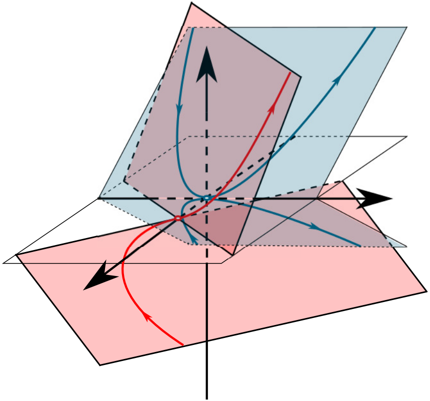

Theorem 3.

a) Each point $\mathbf{p}_\varepsilon$ in $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$ lies in the unfolding of a contact point of order greater than or equal to 2 of the slow subsystem with the fold hyperplane $\mathcal{F}$.

b) If $n = 3$, then the maximal canard point (or faux canard point) of order 1 lies in the unfolding of the two-fold visible-visible (or invisible-invisible) point of the slow subsystem.

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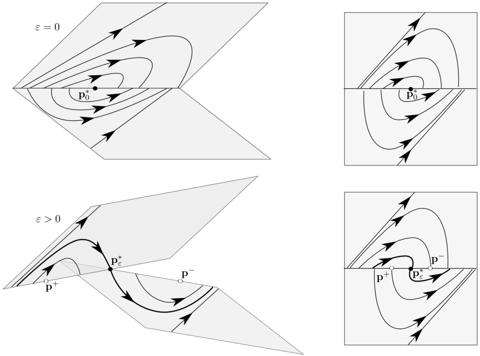

Representation of a 2-dimensional reduced flow. Upper panels: unperturbed case surrounding the invisible two-fold $\mathbf{p}_0^*$. Bottom panels: perturbed flow where the black point $\mathbf{p}_\varepsilon^*$ stands for the faux canard point, while the white points $\mathbf{p}^+$ and $\mathbf{p}^-$ are the breaking points of $\mathbf{p}_0^*$. These white points are invisible two-fold singularities for $\mathcal{S}_\varepsilon^+$ and $\mathcal{S}_\varepsilon^-$.

Conclusions

- R.P., A.E.Teruel, C. Vich
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  Canards, folded nodes and mixed-mode oscillations in piecewise-linear slow-fast systems, SIAM Review, in press, 2016.

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- Introduce a theory for slow-fast dynamics by using PWL systems,
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Idea: reproduce canard-induced MMO behaviour in three-dimensional PWL slow-fast systems and investigate the equivalent of maximal canards (primary, secondary) and folded nodes.

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- Introduce a theory for slow-fast dynamics by using PWL systems,
- and then deriving simplified models that are meaningful for neuroscience applications.

Idea: reproduce canard-induced MMO behaviour in three-dimensional PWL slow-fast systems and investigate the equivalent of maximal canards (primary, secondary) and folded nodes.

Strategies to construct canard type dynamics in 3D PWL

- Canard trajectories in three-dimensional systems (building up on the knowledge from the planar case)

From the planar case, the simplest way to consider three-dimensional models is to put a slow drift on the parameter that displays the canard (or quasi-canard).

For instance[4], for systems in Liénard form

$$\varepsilon \dot{x} = y - f(x), \qquad \dot{y} = a - x. \tag{1}$$

We add a (trivial) slow dynamics on the parameter displaying the explosion in the planar system. Consider the slow drift

$$\dot{a} \quad = \quad c, \quad c \in \mathbb{R}. \tag{2}$$

[4] M. Wechselberger, *Existence and bifurcation of canards in* $\mathbb{R}^3$ *in the case of a folded node*, SIAM Journal on Applied Dynamical Systems, 4 (2005), pp. 101–139.

Strategies to construct canard type dynamics 3D PWL

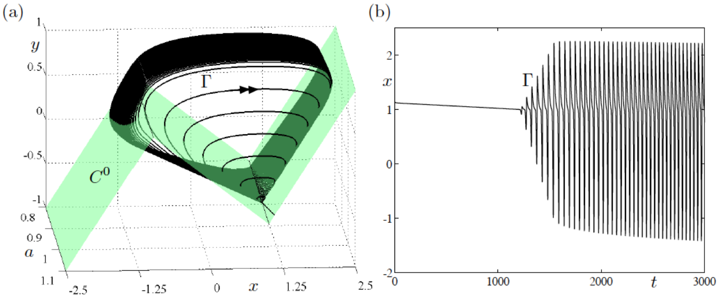
- To approximate a quadratic fold of a smooth slow-fast system, we distinguishing between two-piece local systems and three-piece local systems given by f.

Hence, (1)+(2),

$$\begin{aligned}\varepsilon\dot{x} &= y - f(x),\\ \dot{y} &= a - x,\\ \dot{a} &= c.\end{aligned}$$

Generating mechanism: quasi-canard explosion

$$f(x) = x + \frac{1}{2}(1+k)(|x-1| - |x+1|)$$



Transient MMO in a three-dimensional version of the two-piece local system (1). Parameter values for this transient MMO trajectory are: $\varepsilon = 0.1$, $k = 0.5$, $c = -0.001$.

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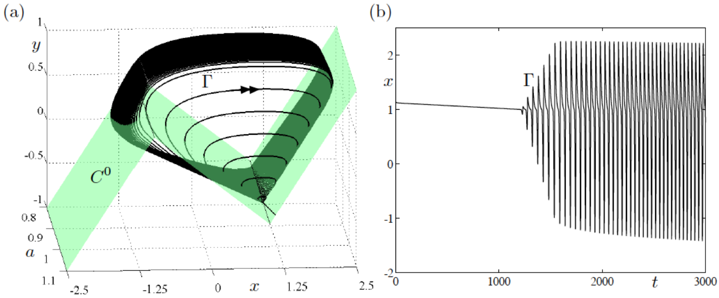

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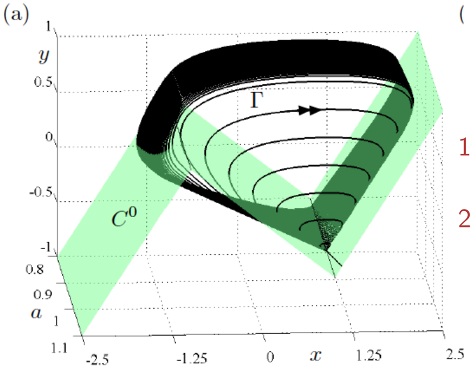

1) an explosive behaviour in the growth of small oscillations
2) no repelling slow manifold

Hence, one can create transient MMO dynamics but not of true canard type.

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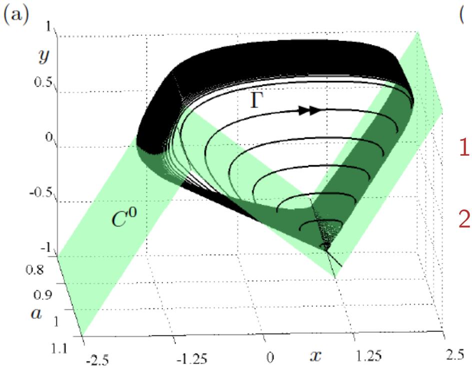


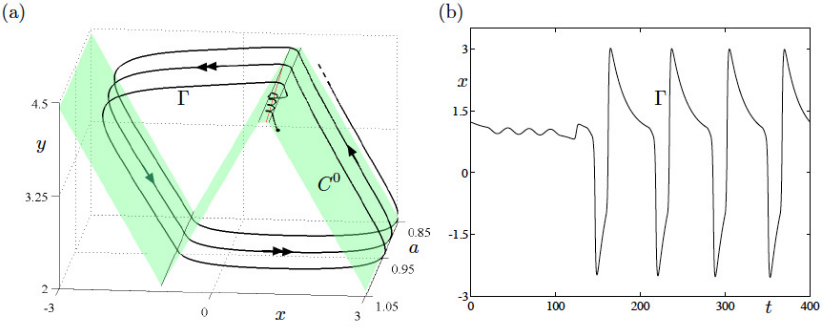
1) an explosive behaviour in the growth of small oscillations
2) no repelling slow manifold

Hence, one can create transient MMO dynamics but not of true canard type.

Generating mechanism: canard explosion

$$f(x) = F_\delta(x) = \begin{cases} -x + (\beta+1)\delta & \text{if} \quad x \geq \delta, \\ \beta x & \text{if} \quad |x| \leq \delta, \\ \ \ x - (\beta-1)\delta & \text{if} \quad x_0 < x < -\delta, \\ -x + 2x_0 - (\beta-1)\delta & \text{if} \quad x \leq x_0. \end{cases}$$

Canard-induced MMOs in transient dynamics exactly as in the smooth case



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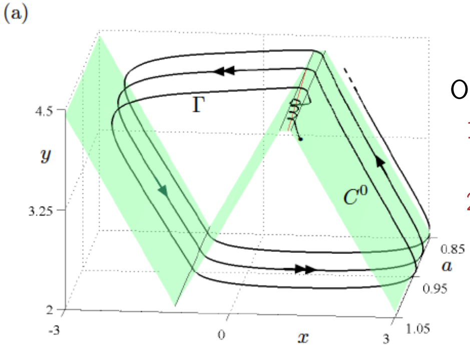


Observe:

1) the four-piece PWL critical manifold
2) a dynamic canard explosion, and hence, folded node type dynamics

The parameter values of the critical manifold are the same as in [AON97], and the speed of the drift is $c = -0.01$.

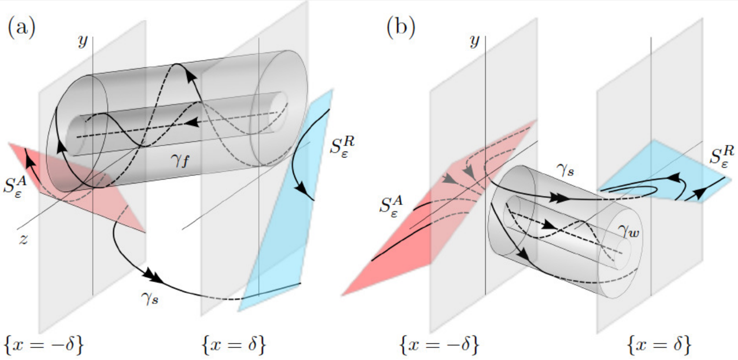
Dynamics near the PWL equivalent of folded singularities

Three-piece local system,

$$\begin{aligned} \varepsilon\dot{x} &= -y + f(x) \\ \dot{y} &= p_1 x + p_2 z \\ \dot{z} &= p_3, \qquad \text{where} \qquad f = f_\delta. \end{aligned}$$

$$f_\delta(x) = \begin{cases} 0 & \text{if} \quad |x| \le \delta, \\ |x| - \delta & \text{if} \quad |x| \ge \delta. \end{cases}$$

Dynamics near the PWL equivalent of folded singularities



Case $p_1 > 0$: (a) folded saddle, (b) folded node .

In the central zone, $H(x,y,z) = \varepsilon p_1(p_1 x + p_2 z)^2 + (p_1 y - \varepsilon p_2 p_3)^2$, is a first integral. It is either a hyperbola ($p_1 < 0$) or a cylinder ($p_1 > 0$), with axis $x = -\frac{p_2}{p_1} z, \quad y = \frac{\varepsilon p_2 p_3}{p_1}$. If $p_1 < 0$ no rotation can happen in this region.

Dynamics near the PWL equivalent of folded singularities

If $p_1 > 0$, the eigenvalues are $\pm i\sqrt{\varepsilon p_1}$, therefore trajectories do rotate in this region.

The line segment organises the dynamics of the full system by acting as an axis of rotation for trajectories that display Small-Amplitude Oscillations (SAOs) in the central zone, which corresponds to the so-called *weak canard* in the smooth case.

It can be proved that the associated maximal winding number μ is obtained as

$$\mu = \frac{\delta}{\pi\sqrt{\varepsilon}} \frac{p_1\sqrt{p_1}}{|p_2 p_3|}$$

Note that μ is reminiscent of the eigenvalue ratio at a folded singularity in the smooth setting

Dynamics near the PWL equivalent of folded singularities

In the smooth case, this maximal winding number is independent of ε.

Thus, in order to reproduce quantitatively the behaviour observed in the smooth context, we choose

$$\delta = \pi\sqrt{\varepsilon},$$

and hence, the maximal winding number is

$$\mu = \frac{p_1\sqrt{p_1}}{|p_2 p_3|}$$

This choice gives a complete match (qualitative and quantitative) with the behaviour of smooth slow-fast systems near folded singularities. That is,

Dynamics near the PWL equivalent of folded singularities

the ε-dependence of δ given by

$$\delta = \pi\sqrt{\varepsilon},$$

forces the central zone collapse to a single corner-line in the singular limit $\varepsilon = 0$, that is, the three-piece local system for $\varepsilon > 0$ converges, in the singular limit, to a two-piece local system. Hence,

- one can see the central zone, needed to obtain canard dynamics, as a blow-up of the corner-line that exists in the singular limit.
- the size of this blow-up, $O(\sqrt{\varepsilon})$, matches that of the smooth case.

Maximal canards and weak canards

But, are there maximal canards? i.e. explicit solutions passing from the attracting slow manifold to the repelling one.

It can be shown that, yes.

In particular it can be shown that, indeed, there is a unique maximal canard, that passes from one side to the other without completing a full rotation; by definition, this special solution is the *primary canard* or strong canard.

There are, also, maximal canards completing full k rotations, for some values of k, named *secondary canards*.

Proposition

Consider $p_3 > 0$, $\delta = \pi\sqrt{\varepsilon}$ and ε small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

a) If $p_1 > 0$ and $p_2 < 0$, for every integer k with $0 \leq k \leq [\mu]$, there exists a maximal canard γ_k intersecting the switching plane $\{x = -\delta\}$ at $\mathbf{p}_k = (-\delta, y_k, z_k)$ where

$$\begin{aligned} y_k &= -\left(\left(k + \frac{1}{2}\right)\frac{p_2 p_3}{\sqrt{p_1}} + p_1\right)\pi\varepsilon^{\frac{3}{2}} - p_2 p_3 \varepsilon^2 + O(\varepsilon^{\frac{5}{2}}), \\ z_k &= -\left(k + \frac{1}{2}\right)\frac{p_3}{\sqrt{p_1}}\pi\sqrt{\varepsilon} + O(\varepsilon). \end{aligned} \tag{3}$$

Moreover, γ_k turns k times around the weak canard γ_w, therefore γ_0 is the strong canard.

Proposition

Consider $p_3 > 0$, $\delta = \pi\sqrt{\varepsilon}$ and ε small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

b) If $p_1 > 0$ and $p_2 > 0$, there exists a unique maximal canard γ_0 intersecting the switching plane at $\mathbf{p}_0 = (-\delta, y_0, z_0)$ where the coordinates y_0 and z_0 satisfy equation (3) with $k = 0$. Since, γ_0 turns less that one time around the faux canard γ_f, therefore γ_0 is the strong canard.

c) If $p_1 < 0$, there are no maximal canards.

Finally, we show on an example how to construct a (linear) global return and obtain PWL MMOs.

A PWL example with MMOs

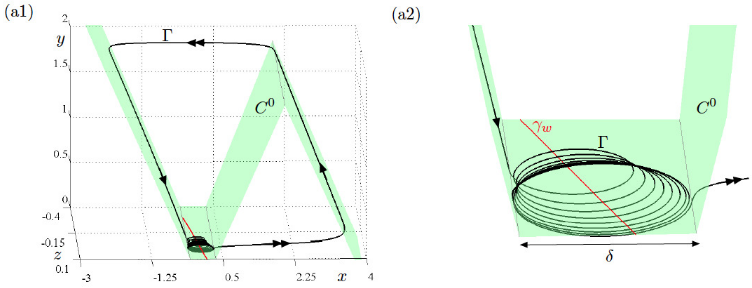
A global return near a PWL folded node, so that we can create canard-induced MMOs.

First, adding a fourth zone to allow for LAOs

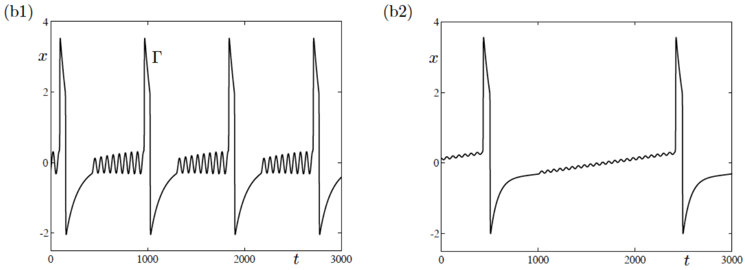
$$\widetilde{f}_\delta(x) = \begin{cases} -x-\delta & \text{if} \quad x \le -\delta, \\ \;\; 0 & \text{if} \quad |x| \le \delta, \\ \;\; x-\delta & \text{if} \quad \delta < x < x_0, \\ -x+2x_0-\delta & \text{if} \quad x \ge x_0. \end{cases}$$

Then, add linear terms to the z equation in order to obtain a global return mechanism.

$$\begin{aligned} \varepsilon \dot{x} &= -y + \widetilde{f}_\delta(x) \\ \dot{y} &= p_1 x + p_2 z \\ \dot{z} &= p_3 + \alpha_1(x-\kappa) + \alpha_2(y-\zeta) + \alpha_3(z-\xi). \end{aligned}$$



Periodic PWL MMO Γ near a folded node. Panels (a1) and (a2) show a phase-space representation of Γ together with the 4-piece PWL critical manifold C^0; panel (a2) is a zoom of panel (a1) near the central flat zone, highlighting the SAOs



Panel (b1) shows the time profile of Γ for the fast variable x. Panel (b2) shows a similar MMO obtained by imposing conditions so that Γ has SAOs with a constant amplitude.

MMO in this model have SAOs with increasing amplitude as the trajectory travels through the central zone. This is simply due to the fact that the eigenvalues in the central zone have non-zero real part because of the new terms in the z-equation.

Quantitative analysis in PWL diff. eq. (Application):

Synaptic conductances estimation in a McKean neuron model.

A. Guillamon, R. P., A.E. Teruel and C. Vich, *Estimation of the synaptic conductance in a McKean-model neuron*. Preprint 2016.

To understand the flow of information in the brain,

estimating the synaptic conductances impinging on a single neuron, directly from its membrane potential, is one of the open problems.

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Simplified model of neuronal activity, namely a piecewise linear version of the Fitzhugh-Nagumo model, the McKean model

$$C\frac{dv}{dt} = f(v) - w - w_0 + I - I_{\mathsf{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0,$$

where f is a 3-zone piecewise linear function,

$$f(v) = \begin{cases} -v & v < a/2, \\ v - a & a/2 \leq v \leq (1+a)/2, \\ 1 - v & v > (1+a)/2. \end{cases}$$

- variables: membrane potential, v, the fast variable and w the slow component,
- parameters: membrane capacitance, C, $0 < C < 0.1$; total current that the neuron is receiving from non-synaptic inputs, I; v_0, w_0, γ and a conductance properties and combinations of membrane reversal potentials.

$$C\frac{dv}{dt} = f(v) - w - w_0 + I - I_{\mathsf{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0,$$

- we consider the synaptic current[5] $I_{\mathsf{syn}} = g_{\mathsf{syn}}(v - v_{\mathsf{syn}})$ apart from the total one
- g_{syn} stands for the synaptic conductance and is considered to be constant

Therefore, I_{syn} can be understood as a representation of the mean field of the synaptic inputs.

[5] Synaptic current is the movement of charge through the postsynaptic membrane due to synaptic transmission. The post-synaptic membrane is the membrane of the nerve after the synapse.

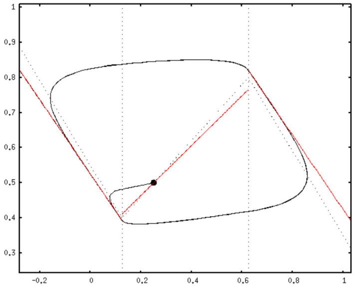
Existence and uniqueness of the periodic orbit

Llibre J, Ordóñez M, Ponce E. *On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry*. (2013). Nonlinear Anal. Real World Appl.

If

$$g_{syn} > 1 - \frac{1}{\gamma}, \quad I_1 < I < I_2 \quad \text{and} \quad |g_{syn} + C\gamma| < 1,$$

Th.1 gives that there exists a limit cycle which is unique and stable.



$\Rightarrow$ At a first step, we infer steady synaptic conductances from the cell's oscillatory activity.

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In practice...

to solve

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we approximate $T(g_{syn})$ and $\widetilde{T}$ by

- T_a an analytical approximation of $T(g_{syn})$
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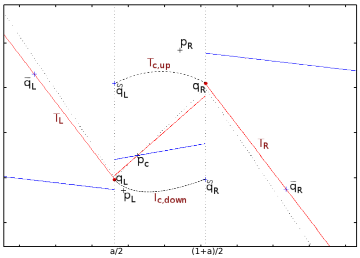
Steady synaptic conductances estimation

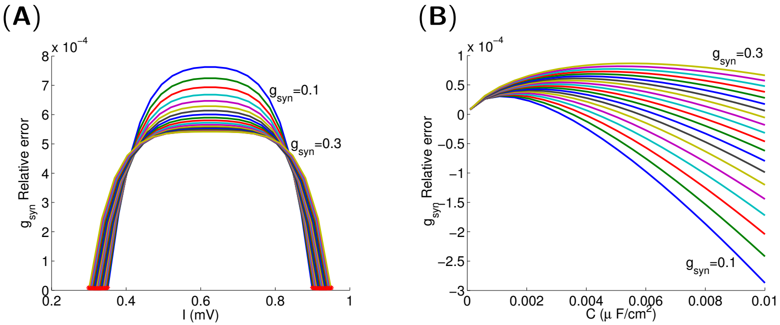


$$T_{lateral} = \frac{1}{2\lambda_s} \ln \left(\left| \frac{\gamma(I - I_i)}{\gamma(I - I_i) - K} \right| \right)$$

$$T_{central} = \frac{1}{2\lambda_q} \ln \left(\left| \frac{\gamma(I - I_i) + K_{1,i}}{\gamma(I - I_i) + K_{2,i}} \right| \right), \quad i = 1, 2.$$

$T_{lateral}$ stands for T_L when $i = 1$ and for T_R when $i = 2$. $T_{central}$ stands for $T_{c,up}$ when $i = 1$ and for $T_{c,down}$ when $i = 2$. K, $K_{1,i}$ and $K_{2,i}$ are functions depending on the system parameters and they have a non-linear dependence with g_{syn}.

Relative error when we estimate the synaptic conductance. Different traces correspond to different values of g_{syn}



(A) versus the applied current, I, for $C = 10^{-4}$.

(B) versus the membrane capacitance, C, for $I = I_1 + 10^{-3}$

$a = 0.25,\ v_0 = 0, \gamma = 0.5,\ w_0 = 0,\ v_{syn} = 0.25 + a/2.$

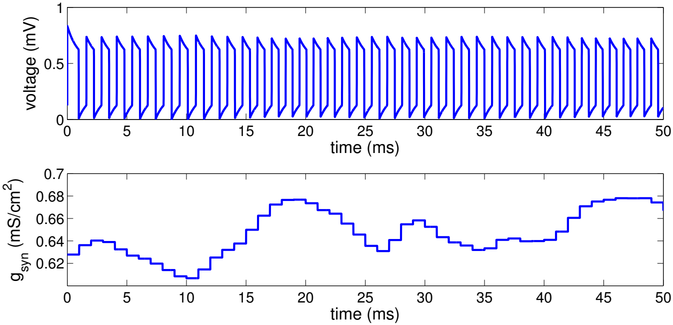
Variable synaptic conductances estimation

We want to estimate g_{syn} when the neuron is regularly spiking.

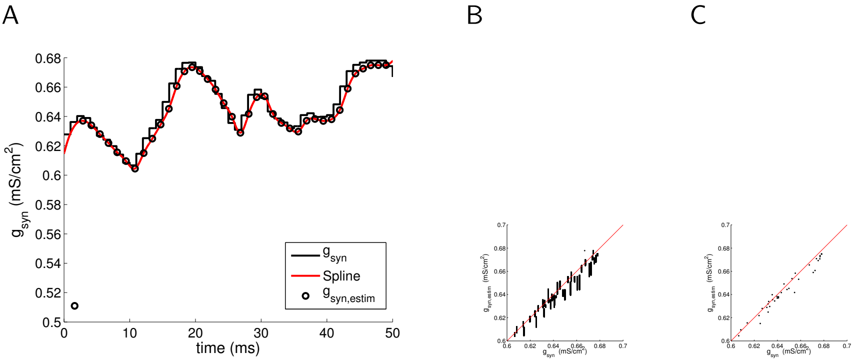
Idea:

1. Solve McKean system using the RK78 method.
2. Once we have $v(t)$, we find the different peaks of $v(t)$ and we compute the differences in time to obtain the sequence of periods $\{T_1, \ldots, T_k\}$.
3. For each T_k we get g_{syn}^k by using the steady synaptic conductance estimation for $T(g_{syn}^k, C, I) = T_k$.
4. We interpolate to obtain $g_{syn}(t)$.

Computational network that models layer $4C\alpha$ of primary visual cortex (McLaughlin et al (2000) and Tao et al (2004)).



Estimation: Panel **A** shows the real and the estimated conductances vs time. The estimation fits the synaptic conductance with a small shift which is larger as C increase. $C = 0.001$.



B and **C**: scatter plot of the real vs the estimated
Panel B: after interpolation; Panel C: only with estimated values.

Parameters: $a = 0.25$, $v_0 = 0$, $\gamma = 0.5$, $w_0 = 0$, $v_{syn} = 0.25 + a/2$, $C = 0.001\ \mu F/cm^2$, $I = 0.625\ \mu A/cm^2$, $g_{syn}(t_0) = 0.6278$.

Thanks for your attention, 谢谢