A Step-type Solution for the Singularly Perturbed Optimal Control Problem

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June 2016

OUTLINE

The problem of contrast structure is a singularly perturbed problem whose solutions with both internal transition layers and boundary layers . The significant feature of the solution is that it will vary rapidly in the thin internal layer. The contrast structure has a strong application background. For example, in the study of physics , there are cases that their solutions vary rapidly in the interior of domain. In recent years, the study of contrast structure is one of the hot research topics in the study of singular perturbation theory[1-3]. More and more scholars begin to pay attention to the contrast structure of variational problem.

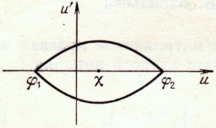

Рис. 1

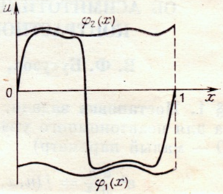

Рис. 2

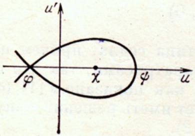

Рис. 3

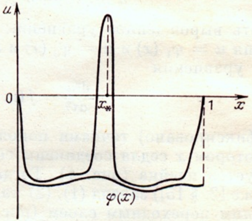

Рис. 4

$$
\begin{cases}
J[u] = \int_0^T (a(y,t) + b(y,t)u + \frac{1}{2}\epsilon^2 u^2)dt \to \min\limits_u, \\
\epsilon y' = u, \quad 0 < \epsilon \ll 1, \\
y(0,\epsilon) = y^0, \quad y(T,\epsilon) = y^T, \quad u \in R.
\end{cases}
$$

M.K. Ni ,G.M. Dmitriev. Contrast structures in a simple vector variational problem and their asymptotic. Automation and Remote Control, 1998, 5:41-52.

The authors not only give the uniformly valid formal asymptotic solution, but also prove that

$$\min_{y} J[y] = \min_{y_0} J_0(y_0) + \sum_{i=1}^{n} \mu^i \min_{y_i} \tilde{J}_i(y_i) + \cdots ,$$

where $\tilde{J}_i(y_i) = J_i(y_i, \tilde{y}_{i-1}, \cdots, \tilde{y}_0)$, $\tilde{y}_k = \arg(\min_{y} \tilde{J}_k(y))$, $k = \overline{0, i-1}$.

$$
\begin{cases}
J[u] = \int_0^T f(y, u, t)dt \to \min\limits_u, \\
\epsilon y' = u, \quad 0 < \epsilon \ll 1, \\
y(0, \epsilon) = y^0, \quad y(T, \epsilon) = y^T \quad u \in R.
\end{cases}
$$

Ni M.K, Vasil'eva, A.B. and Dmitriev M.G. On a steplike contrast structure for a problem of the calculus of variations. Zh. Vychisl. Mat. Mat. Fiz. 2004,44(7):1269-1278 (in Russian) .

Consider the singularly perturbed optimal control problem

$$\begin{cases} J[u] = \int_0^T f(y,u,t)\,\mathrm{d}t \to \min\limits_u, \\ \mu \dfrac{dy}{dt} = a(t)y + b(t)u, \\ y(0,\mu) = y^0, \quad y(T,\mu) = y^T. \end{cases} \tag{1.1}$$

where $\mu > 0$ is a small parameter.

Now, we state the main result in [4], which we will use in the proofs of our main results. Vasil'eva A.B. consider the following boundary value problem

$$\begin{cases} \mu\dfrac{dy}{dt} = F(y, z, t, \mu), \\ \mu\dfrac{dz}{dt} = G(y, z, t, \mu), \\ y(0, \mu) = y^0, \; y(T, \mu) = y^T. \end{cases} \tag{1.2}$$

Suppose that the following assumptions hold

B_1 The reduced system

$$\begin{cases} F(\bar{y}, \bar{z}, t, 0) = 0, \\ G(\bar{y}, \bar{z}, t, 0) = 0, \end{cases}$$

has two isolated roots $(\varphi_1(t), \psi_1(t))$ and $(\varphi_2(t), \psi_2(t))$.

B_2 In the phase plane $(\tilde{y}, \tilde{z})$, the points $M_1(\varphi_1(\bar{t}), \psi_1(\bar{t}))$ and $M_2(\varphi_2(\bar{t}), \psi_2(\bar{t}))$ are stationary saddle points for the associated system

$$\begin{cases} \dfrac{d\tilde{y}}{d\tau} = F(\tilde{y}, \tilde{z}, \bar{t}, 0), \\ \dfrac{d\tilde{z}}{d\tau} = G(\tilde{y}, \tilde{z}, \bar{t}, 0), \end{cases} \tag{1.3}$$

where $\bar{t}$ is a parameter, and system (1.3) has a first integral $\Omega_i(\tilde{y}, \tilde{z}, \bar{t}) = \Omega_i(\varphi_i(\bar{t}), \psi_i(\bar{t}), \bar{t})$, which passes through $M_i, i = 1, 2$.

B_3 The equations $\Omega_i(\tilde{y}, \tilde{z}, \bar{t}) = \Omega_i(\varphi_i(\bar{t}), \psi_i(\bar{t}), \bar{t})$ are solvable with respect to $\tilde{z}$:

$$S_{M_1}: \quad \tilde{z}^{(-)} = V(\tilde{y}, \varphi_1(\bar{t}), \psi_1(\bar{t}), \bar{t}),$$

$$S_{M_2}: \quad \tilde{z}^{(+)} = V(\tilde{y}, \varphi_2(\bar{t}), \psi_2(\bar{t}), \bar{t}).$$

B_4 The equation $H(\bar{t}) = \tilde{z}^{(+)} - \tilde{z}^{(-)}$ has a solution $\bar{t} = t_0 \in (0, T)$, such that $\frac{d}{dt}H(t_0) \neq 0$.

Then boundary value problem (1.2) has a step-like contrast structure solution satisfying the limiting relations

$$\lim_{\mu\to 0} y(t,\mu) = \begin{cases} \varphi_1(t), & t < t_0, \\ \varphi_2(t), & t > t_0. \end{cases}$$

$$\lim_{\mu\to 0} z(t,\mu) = \begin{cases} \psi_1(t), & t < t_0, \\ \psi_2(t), & t > t_0. \end{cases}$$

We will use the main result of [4] to prove our result.The following assumptions are crucial.

A_1 Suppose that the function $f(y,u,t)$ is sufficiently smooth on the domain $D=\{0\leqslant t\leqslant T, |\ y\ |< A\}$, where A is positive constant.

A_2 Suppose that the function $f_{u^2}(y,u,t)>0$ on the domain D .

Formally setting $\mu = 0$ in (1.1), we obtain the reduced problem

$$J[\bar{u}] = \int_0^T f(\bar{y}, \bar{u}, t)\,\mathrm{d}t \to \min_{\bar{u}}, \quad \bar{u} = -b^{-1}(t)a(t)\bar{y}.$$

For our convenience, the above problem can be written in the following equivalent form

$$J[\bar{u}] = \int_0^T F(\bar{y}, t)\mathrm{d}t \to \min_{\bar{y}},$$

where $F(\bar{y}, t) = f(\bar{y}, -b^{-1}(t)a(t)\bar{y}, t)$.

A_3 Suppose that there exist two isolated functions $\bar{y} = \varphi_1(t)$, $\bar{y} = \varphi_2(t)$ such that

$$\min_{\bar{y}} F(\bar{y}, t) = \begin{cases} F(\varphi_1(t), t) & 0 \le t \le t_0, \\ F(\varphi_2(t), t), & t_0 \le t \le T, \end{cases} \tag{1.4}$$

$$\lim_{t \to t_0^-} \varphi_1(t) \neq \lim_{t \to t_0^+} \varphi_2(t).$$

where $F(\bar{y}, t) = f(\bar{y}, -b^{-1}(t)a(t)\bar{y}, t)$.

A_4 Suppose that the transition point t_0 is determined by the equation

$$F(\varphi_1(t_0), t_0) = F(\varphi_2(t_0), t_0),$$

and satisfies the condition

$$\frac{d}{dt} F(\varphi_1(t_0), t_0) \neq \frac{d}{dt} F(\varphi_2(t_0), t_0).$$

It should be noted that the assumptions above is given by the optimal control problem. Next, we will prove that the assumption A_2 is equivalent to the assumption B_2, the assumption A_3 is equivalent to the assumption B_1, the assumption A_4 is equivalent to the assumption B_3 and B_4.

From A_3, it is easy to obtain that

$$\bar{u}(t) = \begin{cases} \alpha_1(t) = -b^{-1}(t)a(t)\varphi_1(t), & 0 \le t < t_0, \\ \alpha_2(t) = -b^{-1}(t)a(t)\varphi_2(t), & t_0 < t \le T, \end{cases}$$

$$\begin{cases} F_y(\varphi_1(t), t) = 0, & F_{yy}(\varphi_1(t), t) > 0, & 0 \le t \le t_0, \\ F_y(\varphi_2(t), t) = 0, & F_{yy}(\varphi_2(t), t) > 0, & t_0 \le t \le T. \end{cases} \tag{1.5}$$

Consider the Hamiltonian function

$$H(y,u,\lambda,t) = f(y,u,t) + \lambda\mu^{-1}[a(t)y + b(t)u],$$

where λ is Lagrange multiplier. The necessary optimality conditions imply that

$$\begin{cases} \mu y' = a(t)y + b(t)u, \\ \lambda' = -f_y(y,u,t) - \lambda\mu^{-1}a(t), \\ \mu f_u(y,u,t) + \lambda(t)b(t) = 0, \\ y(0,\mu) = y^0, \quad y(T,\mu) = y^T. \end{cases} \tag{1.6}$$

From (1.6), we can obtain the following singularly perturbed boundary value problem

$$
\begin{cases}
\mu y' = a(t)y + b(t)u, \\
\mu u' = g_1(y,u,t) + \mu g_2(y,u,t), \\
y(0,\mu) = y^0, \quad y(T,\mu) = y^T,
\end{cases}
\tag{1.7}
$$

where

$$
g_1 = b(t)f_{u^2}^{-1}f_y - a(t)f_{u^2}^{-1}f_u - f_{u^2}^{-1}f_{uy}(a(t)y + b(t)u),
$$

$$
g_2 = b^{-1}(t)b'(t)f_{u^2}^{-1}f_u - f_{u^2}^{-1}f_{ut}.
$$

It is easy to see that the associated system for (1.7) can be written as

$$\begin{cases} \dfrac{du}{d\tau} = b(\bar{t})f_{u^2}^{-1}f_y - a(\bar{t})f_{u^2}^{-1}f_u - f_{u^2}^{-1}f_{uy}(a(\bar{t})y + b(\bar{t})u), \\ \dfrac{dy}{d\tau} = a(\bar{t})y + b(\bar{t})u, \end{cases} \tag{1.8}$$

where $\tau = (t - \bar{t})/\mu, \bar{t} \in [0, T]$ is a parameter.

Lemma

Suppose that A_1 -A_4 hold. Then associated system (1.8) has two equilibria $M_i(\varphi_i(\bar{t}), \alpha_i(\bar{t})),\ i = 1, 2$, which are both saddle points.

Lemma

For fixed $\bar{t} \in [0, T]$, *associated system* (1.8) *has a first integral*

$$(a(\bar{t})y + b(\bar{t})u)f_u(y, u, \bar{t}) - b(\bar{t})f(y, u, \bar{t}) = C, \tag{1.9}$$

where C *is a constant.*

Lemma

Suppose that A_1-A_2 *and* $u \neq -a(\bar{t})b^{-1}(\bar{t})y$ *hold. Then, for fixed* $\bar{t} \in [0, T]$*, the first integral* (1.9) *is solvable with respect to* u*.*

Let

$$H(\bar{t}) = u^{(-)}(0,\bar{t}) - u^{(+)}(0,\bar{t}) = h^{(-)}(y^{(-)}(0),\bar{t},\varphi_1(\bar{t})) - h^{(+)}(y^{(+)}(0),\bar{t},\varphi_2(\bar{t})),$$

where $y^{(-)}(0) = y^{(+)}(0) = \dfrac{1}{2}(\varphi_1(\bar{t}) + \varphi_2(\bar{t})) = \beta(\bar{t})$.

Lemma

Suppose that A_1-A_4 hold. Then $H(t_0) = 0$ if and only if

$$f(\varphi_1(t_0),\alpha_1(t_0),t_0) = f(\varphi_2(t_0),\alpha_2(t_0),t_0).$$

Lemma

Suppose that A_1-A_4 *hold. Then* $\frac{d}{dt}H(t_0) \neq 0$ *if and only if*

$$\frac{d}{dt}f(\varphi_1(t_0), \alpha_1(t_0), t_0) \neq \frac{d}{dt}f(\varphi_2(t_0), \alpha_2(t_0), t_0).$$

Lemma

Suppose that A_1-A_4 *hold. Then there exists* $\bar{t} = t_0$ *at which associated system* (1.8) *has a heteroclinic orbit connecting saddle points* $M_1(\varphi_1(t_0), \alpha_1(t_0))$ *and* $M_2(\varphi_2(t_0), \alpha_2(t_0))$.

Theorem

Suppose that A_1-A_4 hold. Then for sufficiently small $\mu > 0$, the optimal control problem (1.1) has an extremal trajectory $y(t,\mu)$ with a step-like contrast structure

$$\lim_{\mu\to 0} y(t,\mu) = \begin{cases} \varphi_1(t), & 0 \le t < t_0, \\ \varphi_2(t), & t_0 < t \le T. \end{cases}$$

Currently, there are mainly two ways to construct the formal asymptotic solution. The first way is through the boundary function method [6]. Usually, this method is applied to necessary or sufficient optimality conditions. The second alternative is through direct scheme of boundary function method, which consists in a direct expansion of the optimal control problem. we will apply the direct scheme to the singularly perturbed optimal control problem. As a result of the scheme, we get a minimizing control sequence, each new control approximation decreases the performance index of the given problem. It should be noted that the direct scheme not only make it easy to obtain the relations for the high-order approximations, but also show the nature of the optimal control problem.

Construction of asymptotic solution

By means of direct scheme method, an asymptotic solution of problem (1.1) is sought in the form

$$\begin{cases} y(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{y}_k(t) + L_k y(\tau_0) + Q_0^{(-)} y(\tau)), \quad 0 \le t < t^*, \\ u(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{u}_k(t) + L_k u(\tau_0) + Q_0^{(-)} u(\tau)), \end{cases} \tag{3.1}$$

$$\begin{cases} y(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{y}_k(t) + Q_0^{(+)} y(\tau) + R_k y(\tau_1)), \quad t^* < t \le T, \\ u(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{u}_k(t) + Q_0^{(+)} u(\tau) + R_k u(\tau_1)), \end{cases} \tag{3.2}$$

where$\tau_0 = t\mu^{-1}, \quad \tau = (t - t^*)\mu^{-1}, \quad \tau_1 = (t - T)\mu^{-1}.$

From the main results of [5], we have

$$\min_u J[u] = \min_{u_0} J(u_0) + \sum_{i=1}^{n} \mu^i \min_{u_i} \tilde{J}_i(u_i) + \cdots ,$$

where $\tilde{J}_i(u_i) = J_i(u_i, \tilde{u}_{i-1}, \cdots, \tilde{u}_0)$, $\tilde{u}_k = \arg(\min_{u_k} \tilde{J}_k(u_k))$, $k = \overline{0, i-1}$.

Substituting (3.1), (3.2) into (1.1) and equating separately the terms on t, τ_0, τ and τ_1 by the boundary function method, we can obtain a series of variational problems to determine $\{\bar{y}_k(t), \bar{u}_k(t)\}$, $\{L_k y(\tau_0), L_k u(\tau_0)\}$, $\{Q_k^{(\mp)} y(\tau), Q_k^{(\mp)} u(\tau)\}$, $\{R_k y(\tau_1), R_k u(\tau_1)\}$, $k \geq 0$ respectively.

The variational problem to determine the zero-order coefficients of regular terms $\{\bar{y}_0(t),\ \bar{u}_0(t)\}$ are given by

$$\begin{cases} J_0(\bar{u}_0) = \displaystyle\int_0^T f(\bar{y}_0, \bar{u}_0, t)\, dt \to \min_{\bar{u}_0}, \\ a(t)\bar{y}_0 + b(t)\bar{u}_0 = 0. \end{cases}$$

From A_3, we can get

$$\bar{y}_0 = \begin{cases} \varphi_1(t)\ , 0 \le t < t_0, \\ \varphi_2(t)\ , t_0 < t \le T, \end{cases}$$

$$\bar{u}_0 = \begin{cases} \alpha_1(t) = -a(t)b^{-1}(t)\varphi_1(t)\ , 0 \le t < t_0, \\ \alpha_2(t) = -a(t)b^{-1}(t)\varphi_2(t)\ , t_0 < t \le T. \end{cases}$$

The following variational problems to determine $\{Q_0^{(\mp)}y(\tau), Q_0^{(\mp)}u(\tau)\}$ are given by

$$\begin{cases} Q_0^{(\mp)}J = \displaystyle\int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)}f(\varphi_{1,2}(t_0) + Q_0^{(\mp)}y, \alpha_{1,2}(t_0) + Q_0^{(\mp)}u, t_0)\, d\tau \to \min_{Q_0^{(\mp)}u}, \\ \dfrac{d}{d\tau}Q_0^{(\mp)}y = a(t_0)(\varphi_{1,2}(t_0) + Q_0^{(\mp)}y) + b(t_0)(\alpha_{1,2}(t_0) + Q_0^{(\mp)}u), \\ Q_0^{(\mp)}y(0) = \beta(t_0) - \varphi_{1,2}(t_0), \quad Q_0^{(\mp)}y(\mp\infty) = 0, \end{cases} \tag{3.3}$$

where

$$\Delta_0^{(\mp)}f = f(\varphi_{1,2}(t_0) + Q_0^{(\mp)}y, \alpha_{1,2}(t_0) + Q_0^{(\mp)}u, t_0) - f(\varphi_{1,2}(t_0), \alpha_{1,2}(t_0), t_0).$$

Making the substitutions

$$\tilde{y}^{(\mp)} = \varphi_{1,2}(t_0) + Q_0^{(\mp)} y(\tau), \quad \tilde{u}^{(\mp)} = \alpha_{1,2}(t_0) + Q_0^{(\mp)} u(\tau),$$

The problem (3.3) can be rewritten as

$$\begin{cases} Q_0^{(\mp)} J = \displaystyle\int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)} \tilde{f}(\tilde{y}^{(\mp)}(\tau), \tilde{u}^{(\mp)}(\tau), t_0)\, d\tau \to \min_{\tilde{u}^{(\mp)}(\tilde{y}^{(\mp)})}, \\ \dfrac{d\tilde{y}^{(\mp)}}{d\tau} = a(t_0)\tilde{y}^{(\mp)} + b(t_0)\tilde{u}^{(\mp)}, \\ \tilde{y}^{(\mp)}(0) = \beta(t_0), \quad \tilde{y}^{(\mp)}(\mp\infty) = \varphi_{1,2}(t_0). \end{cases} \tag{3.4}$$

Next, we give the equations and their conditions for determining $\{L_0y(\tau_0), L_0u(\tau_0)\}$ and $\{R_0y^*(\tau_1), R_0u^*(\tau_1)\}$ are as follows

$$\begin{cases} L_0J = \displaystyle\int_0^{\infty} \Delta_0 f(\varphi_1(0) + L_0y, \alpha_1(0) + L_0u, 0)\, d\tau_0 \to \min_{\Pi_0 u}, \\ \dfrac{d}{d\tau} L_0y = a(0)(\varphi_1(0) + L_0y) + b(0)(\alpha_1(0) + L_0u), \\ L_0y(0) = y^0 - \varphi_1(0), \quad L_0y(\infty) = 0, \end{cases}$$

where

$$\Delta_0 f = f(\varphi_1(0) + L_0y, \alpha_1(0) + L_0u, 0) - F(\varphi_1(0), \alpha_1(0), 0),$$

and

$$\begin{cases} R_0 J = \displaystyle\int_{-\infty}^{0} \Delta_0 f(\varphi_2(T) + R_0 y, \alpha_2(T) + R_0 u, T)\, d\tau_1 \to \min_{R_0 u}, \\ \dfrac{d}{d\tau_1} R_0 y = a(T)(\varphi_2(T) + R_0 y) + b(T)(\alpha_2(T) + R_0 u), \\ R_0 y(0) = y^T - \varphi_2(T), \quad R_0 y(-\infty) = 0. \end{cases}$$

where $\Delta_0 f = f(\varphi_2(T) + R_0 y, \alpha_2(T) + R_0 u, T) - f(\varphi_2(T), \alpha_2(T), T)$.

Then, we have so far constructed the leading terms

$$\{\bar{y}_0^*(t),\quad \bar{u}_0^*(t)\},\quad \{L_0y^*(\tau_0),\quad L_0u^*(\tau_0)\},$$

$$\{Q_0^{(\mp)}y^*(\tau),\quad Q_0^{(\mp)}u^*(\tau)\},\quad \{R_0y^*(\tau_1),\quad R_0u^*(\tau_1)\}.$$

Additionally, we can obtain the minimum values of the corresponding optimal control problems L_0J^*, $Q_0^{(\mp)}J^*$, R_0J^*:

$$J_0^*(\bar{u}_0) = \int_0^T f(\bar{y}_0^*, \bar{u}_0^*, t)\, dt,$$

$$L_0J^* = \int_{y^0}^{\varphi_1(0)} \frac{\Delta_0^{(\mp)}f(\check{y}^*, \check{u}^*, 0)}{a(0)\check{y}^* + b(0)\check{u}^*}\, d\check{y},$$

where

$$\check{y}^* = \varphi_1(0) + L_0y^*(\tau_0),\quad \check{u} = \alpha_1(0) + L_0u^*(\tau_0),$$

$$Q_0^{(\mp)} J^* = \pm \int_{\varphi_{1,2}(t_0)}^{\beta(t_0)} \frac{\Delta_0^{(\mp)} f(\tilde{y}^{(\mp)*}, \tilde{u}^{(\mp)*}, t_0)}{a(t_0)\tilde{y}^{(\mp)*} + b(t_0)\tilde{u}^{(\mp)*}} \, d\tilde{y},$$

$$R_0 J^* = \int_{\varphi_2(T)}^{y^T} \frac{\Delta_0^{(\mp)} f(\hat{y}^*, \hat{u}^*, T)}{a(T)\hat{y}^* + b(T)\hat{u}^*} \, d\hat{y},$$

where

$$\hat{y}^* = \varphi_2(T) + R_0 y^*(\tau_1), \quad \hat{u}^* = \alpha_2(T) + R_0 u^*(\tau_1).$$

Theorem

Suppose that A_1-A_4 hold. Then for sufficiently small $\mu > 0$ there exists a step-like contrast structure solution $y(t, \mu)$ of the problem (1.1), moreover, the following asymptotic expansion holds

$$y(t,\mu) = \begin{cases} \varphi_1(t) + L_0 y(\tau_0) + Q_0^{(-)} y(\tau) + O(\mu), & 0 \le t < t_0, \\ \varphi_2(t) + R_0 y(\tau_1) + Q_0^{(+)} y(\tau) + O(\mu), & t_0 < t \le T. \end{cases}$$

$$u(t,\mu) = \begin{cases} \alpha_1(t) + L_0 u(\tau_0) + Q_0^{(-)} u(\tau) + O(\mu), & 0 \le t < t_0, \\ \alpha_2(t) + R_0 u(\tau_1) + Q_0^{(+)} u(\tau) + O(\mu), & t_0 < t \le T. \end{cases}$$

It should be noted that the state variable and the control variable functions discussed above are scalar. Next, we will consider the high-dimensional nonlinear singularly perturbed optimal control problem. We not only prove the existence of step-like contrast structure for the singularly perturbed optimal control problem, but also construct asymptotic solution to the optimal controller and optimal trajectory.

Consider the following singularly perturbed optimal control problem

$$\begin{cases} J[u] = \int_0^T f(x, u, t)\,\mathrm{d}t \to \min\limits_u, \\ \varepsilon \dfrac{dx}{dt} = A(x, t) + B(t)u, \\ x(0, \mu) = y^0, \quad x(T, \mu) = y^T, \end{cases} \tag{3.5}$$

where $\varepsilon > 0$ is a small parameter, $x(t) : [0.T] \to R^n$ is the state variable, $u(t) : [0, T] \to R^n$ is the control variable.

Assumption 1. Assume that the functions $f(x, u, t)$, $A(x, t)$ and $B(t)$ are sufficiently smooth on the domain

$$D = \{(x, u, t) | \mid x \mid \leq A, u \in \mathrm{R}, 0 \leq t \leq T\},$$

$B(t)$ is a inverse matrix.

Setting $\epsilon = 0$ in (3.5), we obtain the reduced problem

$$\begin{cases} J[\bar{u}] = \int_0^T \bar{f}(\bar{x}, \bar{u}, t)dt \to \min\limits_{\bar{u}}, \\ 0 = A(\bar{x}, t) + B(t)\bar{u}, \end{cases}$$

For convenience, problem can be written in the equivalent form

$$J[\bar{u}] = \int_0^T \bar{f}(\bar{x}, t)\mathrm{d}t \to \min_{\bar{x}},$$

where $\bar{f}(\bar{x}, t) = f(\bar{x}, -B^{-1}(t)A(\bar{x}, t), t)$.

Assumption 2 Assume that there exist two isolated functions $\bar{y} = \varphi_1(t)$ and $\bar{y} = \varphi_2(t)$ such that

$$\min_{\bar{x}} \bar{f}(\bar{y}, t) = \begin{cases} \bar{f}(\varphi_1(t), t), & 0 \leq t \leq t_0, \\ \bar{f}(\varphi_2(t), t), & t_0 \leq t \leq T, \end{cases}$$

where $\bar{f}(\bar{x}, t) = f(\bar{x}, -B^{-1}(t)A(\bar{x}, t), t)$, $\lim_{t \to t_0^-} \varphi_1(t) \neq \lim_{t \to t_0^+} \varphi_2(t)$..

From the assumption 2, we can obtain that

$$\bar{u}(t) = \begin{cases} \psi_1(t) = -B^{-1}(t)A(\varphi_1(t), t), & 0 \leq t \leq t_0, \\ \psi_2(t) = -B^{-1}(t)A(\varphi_2(t), t), & t_0 \leq t \leq T, \end{cases}$$

Let us introduce the Hamilton function

$$H(y,u,\lambda,t) = f(y,u,t) + \lambda\mu^{-1}(A(x,t) + B(t)u),$$

where λ is Lagrange multiplier .

The auxiliary system of optimal control problem is

$$\begin{cases} J[u] = -\displaystyle\int_{\alpha}^{\beta} (H(\tau) - H(\bar{t})\, \mathrm{d}t \to \min_{u}, \\ \varepsilon \dfrac{d\tilde{x}}{d\tau} = A(\tilde{x},\bar{t}) + B(\bar{t})\tilde{u}, \\ \tilde{x}(\alpha) = \bar{x}\,, \quad x(\beta) = \bar{\bar{x}}, \end{cases} \tag{3.6}$$

where $\tau = \varepsilon^{-1}(t-\bar{t}),\ 0 \le \bar{t} \le T,\ \bar{t}, \bar{x}, \bar{\bar{x}}$ are some fixed numbers.

Writing separately the auxiliary problem of optimal control at the instants of time t=0, $t = t_0$ and $t = T$, then we can get the following optimal control problems to determine asymptotic expansion.

Problem L_0P:

$$\begin{cases} J[u] = -\int_0^{+\infty} (H(\tau_0) - H(0)\,\mathrm{d}t \to \min_u, \\ \varepsilon\dfrac{d\tilde{x}}{d\tau_0} = A(\tilde{x}, 0) + B(0)\tilde{u}, \\ \tilde{x}(0) = x^0, \quad x(+\infty) = \varphi_1(0), \end{cases} \tag{3.7}$$

Problem Q_0P:

$$\begin{cases} J[u] = -\int_{(-\infty)0}^{0(+\infty)} (H(\tau) - H(t_0)\,\mathrm{d}t \to \min\limits_{u}, \\ \varepsilon \dfrac{d\tilde{x}}{d\tau} = A(\tilde{x}, t_0) + B(t_0)\tilde{u}, \\ x(\pm\infty) = \varphi_{1,2}(t_0), \end{cases} \tag{3.8}$$

Problem R_0P:

$$\begin{cases} J[u] = -\int_0^{-\infty} (H(\tau_1) - H(T)\,\mathrm{d}t \to \min\limits_u, \\ \varepsilon \dfrac{d\tilde{x}}{d\tau_1} = A(\tilde{x}, T) + B(T)\tilde{u}, \\ \tilde{x}(0) = x^T, \quad x(-\infty) = \varphi_2(T), \end{cases} \tag{3.9}$$

Assumption 3. Assume that the matrix

$$\begin{pmatrix} \bar{H}_{xx}, & \bar{H}_{xu} \\ \bar{H}_{ux}, & \bar{H}_{uu} \end{pmatrix}.$$

are positive definite, where $\bar{H}_{(.)}$ is calculated at the point $(\varphi_{1,2}(t), \psi_{1,2}(t))$.

Rewrite the boundary condition, we have

$$B_\mu^L = \{(x,u,t) \mid x(0,\mu) = x^0, \quad t = 0\},$$

$$B_\mu^R = \{(x,u,t) \mid x(T,\mu) = x^T, \ t = T\}.$$

Let $S_1 = \phi_1(\varphi_1(t), \psi_1(t), t)$ and $S_2 = \phi_2(\varphi_2(t), \psi_2(t), t)$

Dynamical behavior of the optimal solution

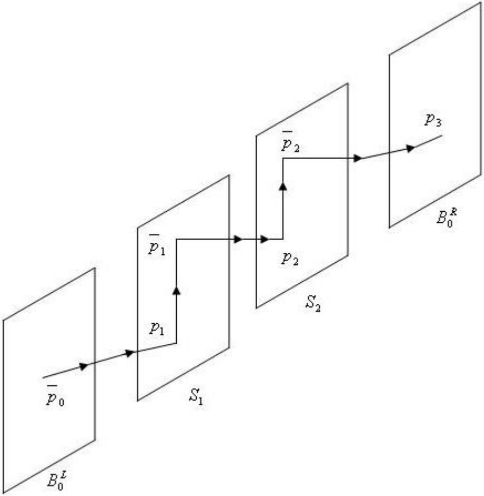

Figure: Singular solution

Singular solution of optimal control system (3.5) is a sequence of solutions to the fast and slow reduced systems corresponding to (3.5) with initial point in B_0^L and final point in B_0^R. Let $\bar{p}_0$ denote the point of the singular solution on B_0^L, p_3 denote the point of the singular solution on B_0^R , p_i and $\bar{p}_i$ denote the point of the singular solution on S_i.

In [7], Belokopytov S.V and Dmitriev M.G consider the singularly perturbed optimal control problem

$$\begin{cases} J[u] = G(x(T), y(T)) + \int_0^T F(x, y, u, t)\,\mathrm{d}t \to \min\limits_u, \\ \dfrac{dx}{dt} = f(x, y, u, t), \\ \varepsilon \dfrac{dy}{dt} = g(x, y, u, t), \\ x(0) = x^0, \quad y(0) = y^0. \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is a small parameter.

Theorem 1 By use of the direct scheme, let us define n control approximations $\tilde{u}_k(t,\varepsilon), k = 0, 1, \cdots, n$. Suppose that the solution $\tilde{x}_k(t,\varepsilon), \tilde{y}_k(t,\varepsilon)$ of (5.1) corresponding to$\tilde{u}_k(t,\varepsilon)$ has an asymptotic representation with the boundary layer. Then for sufficiently small ε

$$J_\varepsilon(\bar{u}_0) \geq J_\varepsilon(\tilde{u}_0) \geq \cdots J_\varepsilon(\tilde{u}_k) \geq J_\varepsilon(\tilde{u}_k + \varepsilon^{k+1}\bar{u}_{k+1}) \geq \cdots J_\varepsilon(\tilde{u}_n)$$

Similarly as the proof of the main result of [7], we can obtain the following theorem.

Theorem

Suppose that the assumptions hold. Then for sufficiently small $\mu > 0$ *there exists a step-like contrast structure solution* $y(t, \mu)$ *of the problem (3.5), moreover, the following asymptotic expansion holds*

$$x(t,\varepsilon) = \begin{cases} \varphi_1(t) + L_0 x(\tau_0) + Q_0^{(-)} x(\tau) + O(\varepsilon), & 0 \le t < t_0, \\ \varphi_2(t) + R_0 x(\tau_1) + Q_0^{(+)} x(\tau) + O(\varepsilon), & t_0 < t \le T. \end{cases}$$

$$u(t,\mu) = \begin{cases} \alpha_1(t) + L_0 u(\tau_0) + Q_0^{(-)} u(\tau) + O(\varepsilon), & 0 \le t < t_0, \\ \alpha_2(t) + R_0 u(\tau_1) + Q_0^{(+)} u(\tau) + O(\varepsilon), & t_0 < t \le T. \end{cases}$$

Example

Consider the problem

$$\begin{cases} J[u] = \int_0^{2\pi} \left(\frac{1}{4}y^4 - \frac{1}{3}y^3 \sin t - y^2 + y \sin t + \frac{1}{2}u^2 \right) \mathrm{d}t \to \min_u, \\ \mu \frac{\mathrm{d}y}{\mathrm{d}t} = -y + u, \\ y(0,\mu) = 0\,, \quad y(2\pi,\mu) = 2. \end{cases} \tag{6.1}$$

Here

$$f(y,u,t) = \frac{1}{4}y^4 - \frac{1}{3}y^3 \sin t - y^2 + y \sin t + \frac{1}{2}u^2.$$

For every t, we have

$$\bar{y}_0(t) = \begin{cases} -1, & 0 \le t < \pi, \\ 1, & \pi < t \le 2\pi. \end{cases}$$

$$\min_{\bar{y}} F(\bar{y}_0, t) = \begin{cases} -\frac{1}{4} - \frac{2}{3}\sin t, & 0 \le t \le \pi, \\ -\frac{1}{4} + \frac{2}{3}\sin t, & \pi \le t \le 2\pi. \end{cases}$$

The transition point $t_0 = \pi$ is determined by the equation $\sin t_0 = 0$.

In this example, the different orbit S_{M_1} and S_{M_2}, which pass through the saddle points $M_1(\bar{t})$ and $M_2(\bar{t})$ respectively, have the form

$$S_{M_1} : u^{(-)} = y^{(-)} + \frac{\sqrt{2}}{2}(1 - y^{(-)2}), \quad S_{M_2} : u^{(+)} = y^{(+)} + \frac{\sqrt{2}}{2}(1 - y^{(+)2}),$$

The left and right zero-order terms of transition layer are determined by the following problems

$$\frac{dQ_0^{(\mp)}y}{d\tau} = -Q_0^{(\mp)}y + Q_0^{(\mp)}u, \quad Q_0^{(\mp)}y(0) = \pm 1, \quad Q_0^{(\mp)}y(\mp\infty) = 0.$$

Its solution are

$$Q_0^{(-)}y = \frac{2e^{\sqrt{2}\tau}}{1+e^{\sqrt{2}\tau}}, \quad Q_0^{(-)}u = \frac{(2+2\sqrt{2}+2e^{\sqrt{2}\tau})e^{\sqrt{2}\tau}}{(1+e^{\sqrt{2}\tau})^2},$$

$$Q_0^{(+)}y = \frac{-2}{1+e^{\sqrt{2}\tau}}, \quad Q_0^{(+)}u = \frac{(2\sqrt{2}e^{\sqrt{2}\tau}-2e^{\sqrt{2}\tau}-2)}{(1+e^{\sqrt{2}\tau})^2}.$$

Similarly, we have

$$L_0y = \frac{2e^{-\sqrt{2}\tau_0}}{1+e^{-\sqrt{2}\tau_0}}, \quad L_0u = \frac{2e^{-\sqrt{2}\tau_0}+2e^{-2\sqrt{2}\tau_0}-2\sqrt{2}e^{-\sqrt{2}\tau_0}}{(1+e^{-\sqrt{2}\tau_0})^2},$$

$$R_0y = \frac{2}{3e^{-\sqrt{2}\tau_1}-1}, \quad R_0u = \frac{6e^{-\sqrt{2}\tau_1}-2+6\sqrt{2}e^{-\sqrt{2}\tau_1}}{(3e^{-\sqrt{2}\tau_1}-1)^2}.$$

Finally, the formal asymptotic solution is

$$y(t,\mu)=\begin{cases}-1+\dfrac{2e^{-\sqrt{2}\tau_0}}{1+e^{-\sqrt{2}\tau_0}}+\dfrac{2e^{\sqrt{2}\tau}}{1+e^{\sqrt{2}\tau}}+O(\mu), & 0\le t<\pi,\\ 1+\dfrac{2}{3e^{-\sqrt{2}\tau_1}-1}+\dfrac{-2}{1+e^{\sqrt{2}\tau}}+O(\mu), & \pi<t\le 2\pi.\end{cases}$$

$$u(t,\mu)=\begin{cases}-1+\dfrac{2e^{-\sqrt{2}\tau_0}+2e^{-2\sqrt{2}\tau_0}-2\sqrt{2}e^{-\sqrt{2}\tau_0}}{(1+e^{-\sqrt{2}\tau_0})^2}\\ +\dfrac{(2+2\sqrt{2}+2e^{\sqrt{2}\tau})e^{\sqrt{2}\tau}}{(1+e^{\sqrt{2}\tau})^2}+O(\mu), 0<t<\pi,\\ 1+\dfrac{6e^{-\sqrt{2}\tau_1}-2+6\sqrt{2}e^{-\sqrt{2}\tau_1}}{(3e^{-\sqrt{2}\tau_1}-1)^2}\\ +\dfrac{(2\sqrt{2}e^{\sqrt{2}\tau}-2e^{\sqrt{2}\tau}-2)}{(1+e^{\sqrt{2}\tau})^2}+O(\mu),\ \pi<t\le 2\pi.\end{cases}$$

Figure: Contrast structure

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Thanks

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