Head-on collision between two hydroelastic solitary waves under an ice sheet

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Overview

Introduction

- Collision process is also known as Interference.
  1. Constructive Interference
  2. Destructive Interference
- Head-on collision is the phenomenon that occurs when two waves meet while traveling along the same medium.
- Linear waves works on the principle of superposition.
- Solitary waves are nonlinear waves that do not obey the principle of superposition.
- Hydroelasticity is related to the deformation of elastic bodies responding to hydrodynamic excitations and concurrently the modification of these excitations owing to the body deformation.

Introduction

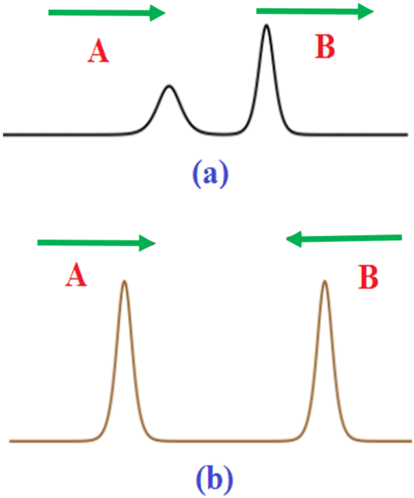


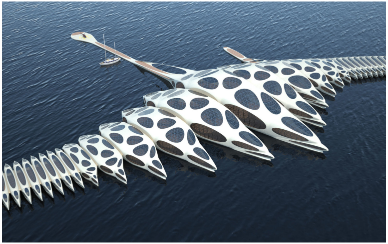
Fig: 1. (a) Overtaking collision, (b) head-on collision

Applications

Fig: 1. Ice Sheet in Antarctica [1]

[1]http://antarcticsun.usap.gov/aroundTheContinent/contentHandler.cfm?id=4108

Applications



(a)

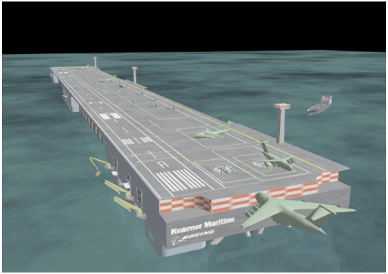
(b)

Fig: 3. (a) Floating hotel[2], (b) Lilypad[3]

[2] http://www.techinsider.io/morphotel-floating-bending-hotel-luxury-2015-12

[3] http://vincent.callebaut.org/page1-img-lilypad.html

Applications



(a)

(b)

Fig: 4. (a) Mobile offshore base[4], (b) Mega float in Tokyo bay[5]

[4] http://www.combatreform.org/itstimetojointheMOB.htm

[5] Tokyo Bay (VLFS), named Mega Float with dimensions of 1000m in length and 120m width.

Mathematical Modeling

Fluid properties

1. Flow is Inviscid
2. Flow is Irrotational
3. Flow is Incompressible
4. Constant density (ρ)
5. Flow is potential (ϕ)

Geometrical properties

1. Cartesian coordinate (x, z)
2. Finite depth $(0 < z < H)$
3. Ice sheet $(z = H)$
4. Thin elastic plate
5. Air effect neglected

Governing Equation and boundary conditions

Laplace equation

$$\nabla^2 \phi = 0, \qquad (0 < z < H). \tag{1}$$

Bottom boundary condition

$$\frac{\partial \phi}{\partial z} = 0, \qquad z = 0. \tag{2}$$

Kinematic boundary condition

$$\frac{\partial H}{\partial t} + \nabla \phi \cdot \nabla H = \frac{\partial \phi}{\partial z}, \qquad z = H. \tag{3}$$

Governing Equation and boundary conditions

Dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + gH + \frac{1}{2}|\nabla \phi|^2 + \frac{D}{\rho}\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right) = B(t). \tag{4}$$

Curvature term

$$\kappa = \frac{H_{xx}}{(1 + H_x^2)^{3/2}}, \tag{5}$$

and

$$H = 1 + \zeta, U = \epsilon(\alpha + \beta), \zeta = \epsilon(\alpha - \beta). \tag{6}$$

where $D = Ed^3/\left[12\left(1-\nu^2\right)\right]$, $B(t)$ is the Bernoulli constant, d is the thickness of the plate, ν is the poisson's ratio, E is Young's Modulus, respectively.

Solution of the Problem

- Poincaré–Lighthill–Kuo[6] (PLK) method.
- This method is Derived from method of strained coordinates by Poincaré in 1892 for ordinary differential equation.
- Later, Lighthill (1949) and Lin (1954) introduced this method for hyperbolic partial differential equations.
- There are three different groups of terms to be integrated in each order of approximation:
    1. Secular terms.
    2. Local terms.
    3. Non-local terms.

To proceed further, we define the following transformations of wave frame coordinates with phase functions

$$\xi = \epsilon^{\frac{1}{2}} K_{\mathrm{R}}(x - C_{\mathrm{R}} t) + \epsilon K_{\mathrm{R}} \theta(\xi, \eta), \quad \eta = \epsilon^{\frac{1}{2}} K_{\mathrm{L}}(x + C_{\mathrm{L}} t) + \epsilon K_{\mathrm{L}} \varphi(\xi, \eta), \tag{7}$$

[6]Van Dyke, M. (1975). Perturbation methods in fluid mechanics/Annotated edition. NASA STI/Recon Technical Report A, 75.

Solution of the Problem

In Eq. (7) ϵ is the dimensionless parameter which describes the amplitude of the wave and order of the magnitude where $(0 < \epsilon \ll 1)$. K_{R}, K_{L} are the wave numbers of left and right going wave of order unity. According to PLK method, we introduce the following expansion

$$\begin{aligned}
\alpha(\xi,\eta) &= \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots \\
\beta(\xi,\eta) &= \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \dots \\
\theta(\xi,\eta) &= \theta_0(\eta) + \epsilon\theta_1(\xi,\eta) + \epsilon^2\theta_2(\xi,\eta) + \dots \\
\varphi(\xi,\eta) &= \varphi_0(\xi) + \epsilon\varphi_1(\xi,\eta) + \epsilon^2\varphi_2(\xi,\eta) + \dots \\
C_{\mathrm{R}} &= 1 + \epsilon a R_1 + \epsilon^2 a^2 R_2 + +\epsilon^3 a^3 R_3 \dots \\
C_{\mathrm{L}} &= 1 + \epsilon b L_1 + \epsilon^2 b^2 L_2 + \epsilon^3 b^3 L_3 \dots
\end{aligned} \tag{8}$$

where $R_1, R_2, R_3, \dots$ and $L_1, L_2, L_3 \dots$ are the parameters for removing secular terms in the perturbation solution.

Solution of the Problem

ξ and η represents the right going and left going phase variables, θ and φ describes the factors regarding phase shifts, a and b are the amplitude factors. we attain the following transformations between derivatives as

$$\frac{\partial}{\partial t} + C_{\mathrm{R}}\frac{\partial}{\partial x} = \frac{\epsilon^{\frac{1}{2}}}{\mathbb{D}}(C_{\mathrm{R}} + C_{\mathrm{L}})\left[K_{\mathrm{L}}\frac{\partial}{\partial \eta} + \epsilon K_{\mathrm{R}}K_{\mathrm{L}}\left(\frac{\partial \theta}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial \theta}{\partial \xi}\frac{\partial}{\partial \eta}\right)\right], \tag{9}$$

$$\frac{\partial}{\partial t} - C_{\mathrm{L}}\frac{\partial}{\partial x} = -\frac{\epsilon^{\frac{1}{2}}}{\mathbb{D}}(C_{\mathrm{R}} + C_{\mathrm{L}})\left[K_{\mathrm{R}}\frac{\partial}{\partial \xi} + \epsilon K_{\mathrm{R}}K_{\mathrm{L}}\left(\frac{\partial \varphi}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial \varphi}{\partial \eta}\frac{\partial}{\partial \xi}\right)\right], \tag{10}$$

where

$$\mathbb{D} = \left(1 - \epsilon K_{\mathrm{R}}\frac{\partial \theta}{\partial \xi}\right)\left(1 - \epsilon K_{\mathrm{L}}\frac{\partial \varphi}{\partial \eta}\right) - \epsilon^2 K_{\mathrm{R}}K_{\mathrm{L}}\frac{\partial \theta}{\partial \eta}\frac{\partial \varphi}{\partial \xi}. \tag{11}$$

Results

(a) Interfacial surface elevation can be written as:

$$\zeta = C_1 A + C_2 A^2 + C_3 B + C_4 B^2 + C_0 AB + \dots, \tag{12}$$

(b) Wave speed:

$$C_R = 1 + C_9 \epsilon_R + C_{10} \epsilon_R^2 + C_{11} \epsilon_R^3 + C_{12} \epsilon_R^4 + \dots, \tag{13}$$

$$C_L = 1 + C_9 \epsilon_L + C_{10} \epsilon_L^2 + C_{11} \epsilon_L^3 + C_{12} \epsilon_L^4 + \dots. \tag{14}$$

Results

(c) Velocity at the bottom (U):

$$U = C_5 A + C_6 A^2 + C_7 B + C_8 B^2 \dots . \tag{15}$$

(d) Phase Shift:

$$\theta = \frac{b}{4K_L} \int_{-\infty}^{\eta} \left[1 + \frac{42A-1}{4} \epsilon a - \frac{13}{4} \epsilon b \right] B \mathrm{d}\eta_1, \tag{16}$$

$$\varphi = \frac{a}{4K_R} \int_{-\infty}^{\xi} \left[1 + \frac{42B-1}{4} \epsilon b - \frac{13}{4} \epsilon a \right] A \mathrm{d}\xi_1, \tag{17}$$

Results

In above Eq. (16) and Eq. (17) the terms which are dependent of ξ and η are just the non-uniform phase shifts i.e. at different points of wave the phase shift is different which causes a distortion in wave during collision. For this purpose the terms which are products of $A(\xi)$ and $B(\eta)$ in Eq. (15) must vanish. Then the right going wave becomes, after setting $B(\eta) = 0$, we have

(e) Distortion Profile:

$$\zeta = C_1 A + C_2 A^2 + \dots . \tag{18}$$

(f) Maximum Run-up Amplitude during Collision: The head-on collision between two solitary waves having maximum height defined by ϵ_R and ϵ_L. It can easily observed when $A(\xi) = 1$ and $B(\eta) = 1$ then maximum run-up amplitude exists at the point ξ and η and hence

$$\begin{aligned}\mathrm{M}_R = \epsilon_L + C_{13}\epsilon_L^3 + C_{14}\epsilon_L^4 + \epsilon_R + C_{13}\epsilon_R^3 + C_{14}\epsilon_R^4 + C_{15}\epsilon_L\epsilon_R + C_{16}\epsilon_L^2\epsilon_R \\ + C_{16}\epsilon_L\epsilon_R^2 + C_{17}\epsilon_L^2\epsilon_R^2.\end{aligned} \tag{19}$$

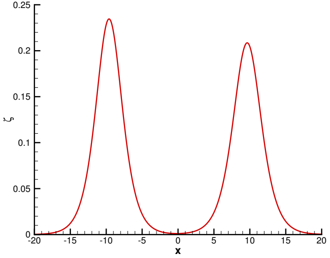
Results

(ii) For two identical solitary waves $\epsilon_R = \epsilon_L$, we get

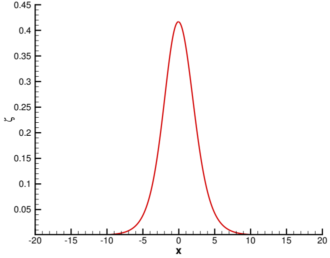
$$\mathrm{M}_R^I = C_{18}\epsilon_L + C_{19}\epsilon_L^2 + C_{20}\epsilon_L^3 + C_{21}\epsilon_L^4. \tag{20}$$

where $A(\xi) = \operatorname{sech}^2 \frac{\xi}{2}$ and $B(\eta) = \operatorname{sech}^2 \frac{\eta}{2}$ with ξ and η, where ξ and η are defined in Eq. (7). The constants $(C_n, n = 1, 2, 3 \ldots)$ appearing in the above equations are arbitrary.

Graphical results



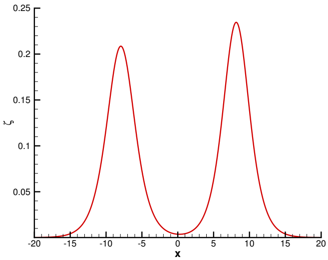
(a) t=t_1



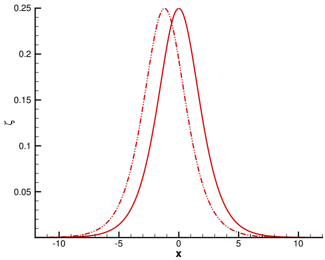
(b) t=t_2

Fig: 5. Head on collision between solitary waves for different time period.

Graphical results



(a) t=t_3



(b)

Fig: 6. (a) Head on collision between solitary waves, (b) Distortion profile. Solid line: Before collision, Dashed line: After collision.

Conclusion

1. Head-on collision between two hydroelastic solitary waves under an ice sheet has been analyzed.
2. Asymptotic solution have been obtained with the help of Poincaré–Lighthill–Kuo (PLK) technique upto third order approximation.
3. During head on collision the maximum wave amplitude occurs at $t = 0$.
4. It is found that after the head on collision, solitary waves regain their original shape and position.
5. It is also observed that wave profile tilts backward after collision in the direction of wave propagation.
6. In Eq. (4) by taking $D \to 0$, the present results reduces to the results obtained by Su and Mirie[7].

[7]Su, C. H., & Mirie, R. M. (1980). On head-on collisions between two solitary waves. J. Fluid Mech. 98(3), 509–525.

Thank you for your attention...
Any comments or suggestion...