Second-Order Slow-Fast System with Piecewise Continuous Term

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Summary

Introduction

Slow-fast system has always been the hot issue of studies of singularly perturbed problems, meanwhile, the **discontinuous/piecewise-continuous dynamical systems** are widely used as model of dynamics, electronics and biology.

The theory of contrast structures, which is an important method of solving classic singularly perturbed problems with more than one isolated root, has been found to be a good method for the problems with piecewise-continuous term. Here we'll seek the asymptotic solution of second-order slow-fast system with Dirichlet boundary value conditions and with piecewise-continuous term by using the method of boundary layer functions and the theory of contrast structures.

Introduction

Consider the following problem:

$$\begin{cases} \mu^2 y'' = F(y,z,t,\mu), \quad z'' = G(y,z,t,\mu), \quad 0 < t < 1, \\ y(0) = y^0, \quad y(1) = y^1, \quad z(0) = z^0, \quad z(1) = z^1, \end{cases} \tag{1}$$

$$F(y,z,t,\mu) = \begin{cases} F_1(y,z,t,\mu), & 0 \le t < t_0, \\ F_2(y,z,t,\mu), & t_0 < t \le 1, \end{cases}$$

while μ is a small parameter, t_0 is given in this problem and satisfies $0 < t_0 < 1$.
F_1, F_2, G are both sufficiently smooth on the set $D = \{(y,z,t,\mu) \| |y| \le l,\ |z| \le l,\ t \in [0,1]\}$, where l is some positive constant, and $F_1(y,z,t_0,0) \neq F_2(y,z,t_0,0)$ for any (y,z).

Assumptions

Condition (1)

The degenerate equation $F(y,z,t,0)=0$ *has an isolated solution*

$$y(t)=\begin{cases}\varphi_1(z,t), & 0\le t\le t_0,\\ \varphi_2(z,t), & t_0\le t\le 1,\end{cases}$$

moreover, while $t=t_0$, $\varphi_1(z,t_0)\neq\varphi_2(z,t_0)$, *and the solutions of following two problems both exist:*

$$\begin{cases}\bar{z}_0^{(\mp)}(t)''=G(\varphi_{1,2}(\bar{z}_0^{(\mp)},t),\bar{z}_0^{(\mp)},t,0),\\ \bar{z}_0^{(-)}(0)=z^0, \quad \bar{z}_0^{(-)}(t_0)=q_0,\\ (\bar{z}_0^{(+)}(t_0)=q_0, \quad \bar{z}_0^{(+)}(1)=z^1).\end{cases}$$

Condition (2)

The following two problems:

$$\begin{cases} y^{(\mp)}(t)'' = [\bar{G}_z(t) - \bar{G}_y(t)\dfrac{\bar{F}_z(t)}{\bar{F}_y(t)}]y^{(\mp)}(t), \\ y^{(-)}(0)(y^{(+)}(t_0)) = 0, \quad y^{(-)}(t_0)(y^{(+)}(1)) = 0, \end{cases} \tag{2}$$

both have trivial solution.

Condition (3)

$F_y(y, z, t, 0) > 0,\ 0 \le t \le 1.$

Methods Involved

Main Thoughts:

- Seek asymptotic solution by the method of boundary layer functions;
- Analyse the existence and exponential decay property of zero-order boundary layer solution by phase plane method and first integral method;
- Describe the internal layer, which is caused by the piecewise-continuous term, through the theory of contrast structures;
- Seek the asymptotic expansion of internal layer in the case of knowing its exact position.

Analysis

Assume the value of solution at point $t = t_0$ are:

$$\begin{cases} y(t_0, \mu) = p(\mu) = p_0 + \mu p_1 + \mu^2 p_2 + \dots, \\ z(t_0, \mu) = q(\mu) = q_0 + \mu q_1 + \mu^2 q_2 + \dots, \end{cases} \tag{3}$$

regard the point $t = t_0$ as boundary, and separate problem (1) into left and right problems, we'll discuss left problem as example, the right one can be discussed similarly:

$$\begin{cases} \mu^2 y^{(-)}(t)'' = F_1(y^{(-)}, z^{(-)}, t, \mu), \quad 0 \le t \le t_0, \\ z^{(-)}(t)'' = G(y^{(-)}, z^{(-)}, t, \mu), \\ y^{(-)}(0, \mu) = y^0, \quad y^{(-)}(t_0, \mu) = p(\mu), \\ z^{(-)}(0, \mu) = z^0, \quad z^{(-)}(t_0, \mu) = q(\mu). \end{cases} \tag{4}$$

Analysis

Construct formal asymptotic solution of left problem:

$$\begin{cases} y^{(-)}(t,\mu) = \bar{y}^{(-)}(t,\mu) + Ly(\tau_1,\mu) + Qy^{(-)}(\xi_1,\mu), \\ z^{(-)}(t,\mu) = \bar{z}^{(-)}(t,\mu) + Lz(\tau_1,\mu) + Qz^{(-)}(\xi_1,\mu), \end{cases} \tag{5}$$

$$\bar{y}^{(-)}(t,\mu) = \sum_{n=0}^{+\infty} \mu^n \bar{y}_n^{(-)}(t), \quad \bar{z}^{(-)}(t,\mu) = \sum_{n=0}^{+\infty} \mu^n \bar{z}_n^{(-)}(t),$$

$$Ly(\tau_1,\mu) = \sum_{n=0}^{+\infty} \mu^n L_n y(\tau_1), \quad Lz(\tau_1,\mu) = \sum_{n=0}^{+\infty} \mu^n L_n z(\tau_1),$$

$$Qy^{(-)}(\xi_1,\mu) = \sum_{n=0}^{+\infty} \mu^n Q_n y^{(-)}(\xi_1), \quad Qz^{(-)}(\xi_1,\mu) = \sum_{n=0}^{+\infty} \mu^n Q_n z^{(-)}(\xi_1),$$

put these into left problem, separate it in different scale, expand it respect to μ and correspond all terms by power of μ, we'll get problem for every term in asymptotic solution.

Analysis

The regular part:

for $k=0$:

$$\begin{cases} 0 = F_1(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0), \quad 0 \le t \le t_0, \\ \bar{z}_0^{(-)}(t)'' = G(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0), \\ \bar{z}_0^{(-)}(0) = z^0, \quad \bar{z}_0^{(-)}(t_0) = q_0. \end{cases} \tag{6}$$

for $k \ge 1$:

$$\begin{cases} \bar{y}_{k-2}^{(-)}(t)'' = \bar{F}_{1y}(t)\bar{y}_k^{(-)} + \bar{F}_{1z}(t)\bar{z}_k^{(-)} + \bar{F}_{1k}(t), \\ \bar{z}_k^{(-)}(t)'' = \bar{G}_y(t)\bar{y}_k^{(-)} + \bar{G}_z(t)\bar{z}_k^{(-)} + \bar{G}_k(t), \\ \bar{z}_k^{(-)}(0) = -L_k z(0), \quad \bar{z}_k^{(-)}(t_0) = q_k - Q_k z^{(-)}(0). \end{cases} \tag{7}$$

Analysis

The right boundary layer part (which is the left part of internal layer, it's similar to the other boundary layers):
for $k = 0$:

$$\begin{cases} \dfrac{\mathtt{d}^2 Q_0 y^{(-)}}{\mathtt{d}\xi_1^2} = F_1(\bar{y}_0^{(-)}(t_0) + Q_0 y^{(-)}, q_0, t_0, 0), \\ \dfrac{\mathtt{d}^2 Q_0 z^{(-)}}{\mathtt{d}\xi_1^2} = 0, \\ Q_0 y^{(-)}(0) = y^0 - \bar{y}_0^{(-)}(0), \quad Q_0 y^{(-)}(+\infty) = 0, \\ Q_0 z^{(-)}(0) = 0, \quad Q_0 z^{(-)}(+\infty) = 0, \end{cases} \tag{8}$$

Analysis

$Q_0 z^{(-)}(\xi_1) \equiv 0,$ so we have the new form:

$$\begin{cases} \dfrac{\mathrm{d}^2 Q_0 y^{(-)}}{\mathrm{d}\xi_1^2} = F_1(\bar{y}_0^{(-)}(t_0) + Q_0 y^{(-)}, q_0, t_0, 0), \\ Q_0 y^{(-)}(0) = p_0 - \bar{y}_0^{(-)}(t_0), \quad Q_0 y^{(-)}(-\infty) = 0, \end{cases} \tag{9}$$

set $\tilde{y}(\xi_1) = \bar{y}_0^{(-)}(t_0) + Q_0 y^{(-)}(\xi_1),\ \hat{u}_1 = \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\xi_1}$, then problem (9) can be:

$$\begin{cases} \dfrac{\mathrm{d}\hat{u}_1}{\mathrm{d}\xi_1} = F_1(\tilde{y}, q_0, t_0, 0), \quad \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\xi_1} = \hat{u}_1, \\ \tilde{y}(0) = p_0, \quad \tilde{y}(-\infty) = \bar{y}_0^{(-)}(t_0). \end{cases} \tag{10}$$

Analysis

For $k \geq 1$ $\left(Q_{-1}y^{(-)} = 0,\ Q_{-1}z^{(-)} = 0 \right)$:

$$\begin{cases} \dfrac{\mathrm{d}^2 Q_k y^{(-)}}{\mathrm{d}\xi_1^2} = \tilde{F}_{1y}(\xi_1) Q_k y^{(-)} + \tilde{F}_{1z}(\xi_1) Q_k z^{(-)} + Q_k F_1^{(-)}(\xi_1), \\ \dfrac{\mathrm{d}^2 Q_k z^{(-)}}{\mathrm{d}\xi_1^2} = \tilde{G}_y(\xi_1) Q_{k-2} y^{(-)} + \tilde{G}_z(\xi_1) Q_{k-2} z^{(-)} + Q_k G^{(-)}(\xi_1), \\ Q_k y^{(-)}(0) = p_k - \bar{y}_k^{(-)}(t_0), \quad Q_k y^{(-)}(-\infty) = 0, \\ Q_k z^{(-)}(0) = q_k - \bar{z}_k^{(-)}(t_0), \quad Q_k z^{(-)}(-\infty) = 0, \end{cases} \tag{11}$$

where $\tilde{f}(\xi_1) = f(\bar{y}_0^{(-)}(t_0) + Q_0 y^{(-)}(\xi_1), q_0, t_0, 0)$,

Analysis

the solution has the following form:

$$Q_k z^{(-)}(\xi_1) = \int_{-\infty}^{\xi_1}\int_{-\infty}^{\eta}[\tilde{G}_y Q_{k-2} y^{(-)} + \tilde{G}_z Q_{k-2} z^{(-)} + Q_k G^{(-)}]\mathtt{d}s\mathtt{d}\eta,$$

$$Q_k y^{(-)}(\xi_1) = (p_k - \bar{y}_k^{(-)}(t_0))\frac{\hat{u}_1(\xi_1)}{\hat{u}_1(0)}$$
$$+ \quad \hat{u}_1(\xi_1)\int_0^{\xi_1}\hat{u}_1^{-2}(\eta)\int_{-\infty}^{\eta}\hat{u}_1(s)(F_{1z}(s)Q_k z^{(-)}(s) + Q_k F^{(-)}(s))\mathtt{d}s\mathtt{d}\eta$$

where $\xi_1 \le 0$.

Analysis

Below, there are some mainly results we need for proof:
The right part of internal layer has the form($\xi_1 \geq 0$):

$$Q_k z^{(+)}(\xi_1) = \int_{+\infty}^{\xi_1} \int_{+\infty}^{\eta} [\tilde{G}_y Q_{k-2} y^{(+)} + \tilde{G}_z Q_{k-2} z^{(+)} + Q_k G^{(+)}] \mathrm{d}s\mathrm{d}\eta,$$

$$Q_k y^{(+)}(\xi_1) = (p_k - \bar{y}_k^{(+)}(t_0)) \frac{\hat{u}_2(\xi_1)}{\hat{u}_2(0)}$$

$$+ \quad \hat{u}_2(\xi_1) \int_0^{\xi_1} \hat{u}_2^{-2}(\eta) \int_{+\infty}^{\eta} \hat{u}_2(s)(F_{2z}(s) Q_k z^{(+)}(s) + Q_k F^{(+)}(s)) \mathrm{d}s\mathrm{d}\eta$$

where $\hat{f}(\xi_1) = f(\bar{y}_0^{(+)}(t_0) + Q_0 y^{(+)}(\xi_1), q_0, t_0, 0)$,
$\hat{y}(\xi_1) = \bar{y}_0^{(+)}(t_0) + Q_0 y^{(+)}(\xi_1), \quad \hat{u}_2 = \dfrac{\mathrm{d}\hat{y}}{\mathrm{d}\xi_1}$.

Analysis

Determination and proof of internal layer:

$$H_1(p_0) = \sqrt{2}[\int_{\bar{y}_0^{(-)}(t_0)}^{p_0} F_1(y, q_0, t_0, 0)\mathtt{d}y]^{\frac{1}{2}} - \sqrt{2}[\int_{\bar{y}_0^{(+)}(t_0)}^{p_0} F_2(y, q_0, t_0, 0)\mathtt{d}y]^{\frac{1}{2}},$$

$$\frac{\mathtt{d}}{\mathtt{d}p_0} H_1(p_0) = \frac{F_1(p_0, q_0, t_0, 0) - F_2(p_0, q_0, t_0, 0)}{\sqrt{\int_{\bar{y}_0^{(-)}(t_0)}^{p_0} F_1(p_0, q_0, t_0, 0)\mathtt{d}y}} \neq 0.$$

Analysis

Set

$$H_2(q_0) = (\bar{z}_0^{(-)})'(t_0, q_0) - (\bar{z}_0^{(+)})'(t_0, q_0),$$

and q_0 can be determined by equation $H_2(q_0) = 0$.

Condition (4)

The solution of $H_2(q_0) = 0$ *exists and* $q_0 \in (\bar{z}_0^{(-)}(t_0), \bar{z}_0^{(+)}(t_0))$, *moreover,*

$$\frac{d}{dq_0} H_2(q_0) = \frac{d}{dq_0}[(\bar{z}_0^{(-)})'(t_0, q_0) - (\bar{z}_0^{(+)})'(t_0, q_0)] \neq 0.$$

Analysis

$$\begin{cases} \bar{y}_{k-1}^{(-)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\xi_1} Q_k y^{(-)}(0) = \bar{y}_{k-1}^{(+)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\xi_1} Q_k y^{(+)}(0), \quad (1) \\ \bar{z}_{k}^{(-)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\xi_1} Q_{k+1} z^{(-)}(0) = \bar{z}_{k}^{(+)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\xi_1} Q_{k+1} z^{(+)}(0), \quad (2) \end{cases}$$

$$\begin{aligned} p_k = {} & [F_1(p_0, q_0, t_0, 0) - F_2(p_0, q_0, t_0, 0)]^{-1}[F_1(p_0, q_0, t_0, 0)\bar{y}_k^{(-)}(t_0) \\ & - F_2(p_0, q_0, t_0, 0)\bar{y}_k^{(+)}(t_0) - \int_{-\infty}^{0} \hat{u}_1(s) f^{(-)}(s)\mathtt{d}s + \int_{+\infty}^{0} \hat{u}_2(s) f^{(+)}(s) \\ & - \hat{u}_1(0)\bar{y}_{k-1}^{(-)}(t_0)' - \hat{u}_1(0)\bar{y}_{k-1}^{(+)}(t_0)']. \end{aligned}$$

Analysis

let $A = \dfrac{\mathtt{d}Q_{k+1}z^{(+)}(0)}{\mathtt{d}\xi_1} - \dfrac{\mathtt{d}Q_{k+1}z^{(-)}(0)}{\mathtt{d}\xi_1}$, it is obvious that A is a constant.

Rewrite the equation in last page:

$$\bar{z}_k^{(-)}(t_0)' - \bar{z}_k^{(+)}(t_0)' = A,$$

after some simplifications and proofs, we can obtain:

$$q_k = [\frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(-)}(t_0)' - \frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(+)}(t_0)']^{-1}A.$$

Main Result

Theorem

The asymptotic solution of problem we talked exists while all the conditions above are met, and the solution has the following expression:

$$y(t,\mu)=\begin{cases} y^{(-)}(t,\mu)=\sum\limits_{k=0}^{n+1}\mu^k(\bar{y}_k^{(-)}(t)+L_k y(\tau_1)+Q_k y^{(-)}(\xi_1))+O(\mu^{n+1}), & 0\le t\le t_0,\\ y^{(+)}(t,\mu)=\sum\limits_{k=0}^{n+1}\mu^k(\bar{y}_k^{(+)}(t)+R_k y(\tau_2)+Q_k y^{(+)}(\xi_1))+O(\mu^{n+1}), & t_0\le t\le 1,\end{cases}$$

$$z(t,\mu)=\begin{cases} z^{(-)}(t,\mu)=\sum\limits_{k=0}^{n+1}\mu^k(\bar{z}_k^{(-)}(t)+L_k z(\tau_1)+Q_k z^{(-)}(\xi_1))+O(\mu^{n+1}), & 0\le t\le t_0,\\ z^{(+)}(t,\mu)=\sum\limits_{k=0}^{n+1}\mu^k(\bar{z}_k^{(+)}(t)+R_k z(\tau_2)+Q_k z^{(+)}(\xi_1))+O(\mu^{n+1}), & t_0\le t\le 1.\end{cases}$$

Example

Consider the following problem:

$$\begin{cases} \mu^2 y'' = F(y, z, t, \mu), \quad z'' = y + z, \quad 0 \le t \le 1, \\ y(0) = -1, \quad y(1) = 2, \quad z(0) = 0, \quad z(1) = 1, \end{cases}$$

where

$$F(y, z, t, \mu) = \begin{cases} y + z + 2, & 0 \le t \le \frac{1}{2}, \\ y + z - 2, & \frac{1}{2} \le t \le 1. \end{cases}$$

Example

Step 1:

$$\varphi_1(z) = -z - 2,\ \varphi_2(z) = 2 - z.$$

Step 2:

$$\bar{z}_0^{(-)}(t)'' = -2, \quad \bar{z}_0^{(-)}(0) = 0, \quad \bar{z}_0^{(-)}(\frac{1}{2}) = q_0, \tag{12}$$

$$\bar{z}_0^{(+)}(t)'' = 2, \quad \bar{z}_0^{(+)}(\frac{1}{2}) = q_0, \quad \bar{z}_0^{(+)}(1) = 1, \tag{13}$$

solve problems (12)-(13), one can obtain:

$$\begin{cases} \bar{z}_0^{(-)}(t) = -t^2 + (2q_0 + \frac{1}{2})t, \\ \bar{z}_0^{(+)}(t) = t^2 + (\frac{1}{2} - 2q_0)t + 2q_0 - \frac{1}{2}, \end{cases}$$

Example

moreover, $q_0 = \frac{1}{2}$, hence

$$\begin{cases} \bar{z}_0^{(-)}(t) = -t^2 + \frac{3}{2}t, \\ \bar{z}_0^{(+)}(t) = t^2 - \frac{1}{2}t + \frac{1}{2}, \end{cases}$$

therefore,

$$\begin{cases} \bar{y}_0^{(-)}(t) = t^2 - \frac{3}{2}t + 2, \\ \bar{y}_0^{(+)}(t) = -t^2 + \frac{1}{2}t + \frac{3}{2}. \end{cases}$$

Example

Step 3:

$$\frac{d^2 L_0 y}{d\tau_1^2} = L_0 y, \quad L_0 y(0) = 1, \quad L_0 y(+\infty) = 0, \tag{14}$$

$$\frac{d^2 R_0 y}{d\tau_1^2} = R_0 y, \quad R_0 y(0) = 1, \quad R_0 y(-\infty) = 0, \tag{15}$$

solve problems (14)-(15) we have $L_0 y(\tau_1) = e^{-\tau_1}$, $R_0 y(\tau_2) = e^{\tau_2}$.

Example

Step 4:

$$\frac{d^2Q_0y^{(-)}}{d\xi_1^2} = Q_0y^{(-)}, \quad Q_0y^{(-)}(0) = p_0 + \frac{5}{2}, \quad Q_0y^{(-)}(-\infty) = 0, \tag{16}$$

$$\frac{d^2Q_0y^{(+)}}{d\xi_1^2} = Q_0y^{(+)}, \quad Q_0y^{(+)}(0) = p_0 - \frac{3}{2}, \quad Q_0y^{(+)}(+\infty) = 0, \tag{17}$$

from problems (16)-(17) we can know

$Q_0y^{(-)}(\xi_1) = (p_0 + \frac{5}{2})e^{\xi_1}$, $Q_0y^{(+)}(\xi_1) = (p_0 - \frac{3}{2})e^{-\xi_1}$.

By virtue of condition of smoothness $p_0 = -\frac{1}{2}$.

Therefore $Q_0y^{(-)}(\xi_1) = 2e^{\xi_1}$, $Q_0y^{(+)}(\xi_1) = -2e^{-\xi_1}$.

Example

Step 5: The solution of example has the following form:

$$y(t,\mu)=\begin{cases} t^2-\dfrac{3}{2}t-2+e^{-\frac{t}{\mu}}+2e^{\frac{t-\frac{1}{2}}{\mu}}+O(\mu), & 0\le t\le \dfrac{1}{2},\\ -t^2+\dfrac{1}{2}t+\dfrac{3}{2}+e^{\frac{t-1}{\mu}}-2e^{\frac{\frac{1}{2}-t}{\mu}}+O(\mu), & \dfrac{1}{2}\le t\le 1,\end{cases}$$

$$z(t,\mu)=\begin{cases} -t^2+\dfrac{3}{2}t+O(\mu), & 0\le t\le \dfrac{1}{2},\\ t^2-\dfrac{1}{2}t+\dfrac{1}{2}+O(\mu), & \dfrac{1}{2}\le t\le 1.\end{cases}$$

Thank You!