# Second Order Quasi-Linear Dirichlet Problem with Discontinuous Righthand Side

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# Outline

- Introduction
- Assumptions
- Construction of Asymptotic Solution
- Main results

# Introduction

- Discontinuous dynamical system is one of the basic instruments to understand better the role of discontinuous in the real world.
- Extending non-continuous theory to singularly perturbed system is a contemporary problem. The nearly achievement in this field can be found in Mingkang Ni & Nefedov N 2015.

Consider the following problem

$$\begin{cases} \varepsilon u'' = A(u,x)u' + f(u,x), & 0 < x < 1, \\ u(0,\varepsilon) = u^0, & u(1,\varepsilon) = u^1, \end{cases} \tag{1.1}$$

where $\varepsilon$ is a small positive parameter, $u$ is scalar-valued state variable.

Moreover, $A$ and $f$ are both discontinuous.

$$A(u,x) = \begin{cases} A^{(-)}(u,x), & 0 \leq x < x_0, \\ A^{(+)}(u,x), & x_0 < x \leq 1, \end{cases} \tag{1.2}$$

$$f(u,x) = \begin{cases} f^{(-)}(u,x), & 0 \leq x < x_0, \\ f^{(+)}(u,x), & x_0 < x \leq 1, \end{cases} \tag{1.3}$$

where $x_0$ is a given number such that $0 < x_0 < 1$.

This question under the continuous assumption has proved that its solution belongs to a $C^2$ function family.

**Definition**

*A function*

$$u(x,\varepsilon) \in C^1[0,1] \cap (C^2(0,x_0) \cup C^2(x_0,1)),$$

*satisfies problem* (1.1) *is the solution of this problem.*

# Assumptions

**Condition 1**

$A^{(-)}, f^{(-)} \in C^{\infty}(\overline{D}_1, \mathbb{R})$, $A^{(+)}, f^{(+)} \in C^{\infty}(\overline{D}_2, \mathbb{R})$, *and the following inequality holds*

$$A^{(-)}(u, x_0) \neq A^{(+)}(u, x_0), \quad u \in I_u,$$

$$f^{(-)}(u, x_0) \neq f^{(+)}(u, x_0), \quad u \in I_u.$$

**Condition 2**

$A^{(-)}$ *and* $A^{(+)}$ *satisfy the inequalities*

$$\begin{cases} A^{(-)}(u, x) > 0, & 0 \leq x \leq x_0, \\ A^{(+)}(u, x) < 0, & x_0 \leq x \leq 1. \end{cases} \tag{2.1}$$

# Assumptions

In order to construct a multi-scale uniformly valid asymptotic solution, let us consider the degenerate and associated systems. The degenerate equation of this problem on the interval $[0, x_0]$

$$\begin{cases} A^{(-)}(u^{(-)}, x)u'^{(-)} + f^{(-)}(u^{(-)}, x) = 0, \quad 0 < x \leq x_0, \\ u^{(-)}(0, \varepsilon) = u^0. \end{cases} \tag{2.2}$$

Similarly the degenerate equation on the interval $[x_0, 1]$

$$\begin{cases} A^{(+)}(u^{(+)}, x)u'^{(+)} + f^{(+)}(u^{(+)}, x) = 0, \quad x_0 \leq x < 1, \\ u^{(+)}(1, \varepsilon) = u^1. \end{cases} \tag{2.3}$$

# Assumptions

For the degenerate equations, we make the assumption:

**Condition 3**

*The degenerate equations* (2.2) *and* (2.3) *have only one solution* $\varphi_1(x) \in C^2(0, x_0)$ *and* $\varphi_2(x) \in C^2(x_0, 1)$.

To be definite, we assume that $\varphi_1(x_0) < \varphi_2(x_0)$.

To find the leading term of the inner transition layer, we introduce the auxiliary variable $z = du/dx$ and suppose that $z^{(\mp)}(x_0, \varepsilon) = \varepsilon^{-1} z_{-1}^{(\mp)} + z_0^{(\mp)} + \varepsilon z_1^{(\mp)} + \cdots$, where $z_i^{(\mp)}$ will be determined later. Then we get the associated system

$$\begin{cases} \dfrac{d\tilde{z}}{d\xi} = A^{(\mp)}(\tilde{u}, x_0)\tilde{z}, \quad \dfrac{d\tilde{u}}{d\xi} = \tilde{z}, \quad \xi \in \mathbb{R}^{\mp}, \\ \tilde{u}(0) = p, \quad \tilde{u}(\mp\infty) = \varphi_i(x_0), \\ \tilde{z}(0) = z_{-1}^{(\mp)}, \quad \tilde{z}(\mp\infty) = 0, \end{cases} \tag{2.4}$$

where $i = 1, 2$, $\xi = (x - x_0)/\varepsilon$ and $p \in (\varphi_1(x_0), \varphi_2(x_0))$.

# Assumptions

By Condition 2 and Condition 3, we have that in the phase plane $(\tilde{u}, \tilde{z})$ there exists a separatrix of the form

$$\Omega^{(\mp)} : \tilde{z} = \int_{\varphi_i(x_0)}^{\tilde{u}} A^{(\mp)}(s, x_0) ds,$$

which enters the equilibrium point $(\varphi_i(x_0), 0)$ as $\xi \to \mp\infty (i = 1, 2)$.

**Condition 4**

*In the phase plane $(\tilde{u}, \tilde{z})$, the vertical $\tilde{u} = p$ intersects the separatrics $\Omega^{(\mp)}$ for any $p \in (\varphi_1(x_0), \varphi_2(x_0))$.*

To construct the asymptotic solution of problem (1.1), we consider two boundary value problems:

$$\begin{cases} \varepsilon u''^{(\mp)} = A^{(\mp)}(u^{(\mp)}, x)u'^{(\mp)} + f^{(\mp)}(u^{(\mp)}, x), \\ u^{(\mp)}(i, \varepsilon) = u^i, \quad u^{(-)}(x_0, \varepsilon) = p(\varepsilon), \quad i = 0, 1, \end{cases} \tag{3.1}$$

The function $p(\varepsilon)$ will be determined later and it has an asymptotic expansion of the form

$$p(\varepsilon) = p_0 + \varepsilon p_1 + \cdots + \varepsilon^k p_k + \cdots , \tag{3.2}$$

where $p_i$ will be determined later.

# Construction of Asymptotic Solution

Making the changes of variable $z^{(\mp)} = du^{(\mp)}/dx$, and a **formal solution** of problem (3.1)

$$u^{(\mp)}(x,\varepsilon) = \bar{u}^{(\mp)}(x,\varepsilon) + Q^{(\mp)}u(\xi,\varepsilon), \tag{3.3}$$

$$z^{(\mp)}(x,\varepsilon) = \bar{z}^{(\mp)}(x,\varepsilon) + Q^{(\mp)}z(\xi,\varepsilon), \tag{3.4}$$

where

$$\bar{u}^{(\mp)}(x,\varepsilon) = \bar{u}_0^{(\mp)}(x) + \varepsilon\bar{u}_1^{(\mp)}(x) + \varepsilon^2\bar{u}_2^{(\mp)}(x) + \cdots,$$

$$\bar{z}^{(\mp)}(x,\varepsilon) = \bar{z}_0^{(\mp)}(x) + \varepsilon\bar{z}_1^{(\mp)}(x) + \varepsilon^2\bar{z}_2^{(\mp)}(x) + \cdots,$$

$$Q^{(\mp)}u(\xi,\varepsilon) = Q_0^{(\mp)}u(\xi) + \varepsilon Q_1^{(\mp)}u(\xi) + \varepsilon^2 Q_2^{(\mp)}u(\xi) + \cdots,$$

$$Q^{(\mp)}z(\xi,\varepsilon) = \varepsilon^{-1}Q_{-1}^{(\mp)}z(\xi) + Q_0^{(\mp)}z(\xi) + \varepsilon Q_1^{(\mp)}z(\xi) + \cdots,$$

$$\xi = (x - x_0)/\varepsilon.$$

# Construction of Asymptotic Solution

In the standard manner, firstly substituting (3.3) and (3.4) into problem (3.1), then separating the variables according to the scale of $x$ and $\xi$, finally equating the coefficients of same power of $\varepsilon$.

**Regular part**

For $k = 0$:

$$\begin{cases} A^{(\mp)}(\bar{u}_0^{(\mp)}, x)\dfrac{d\bar{u}_0^{(\mp)}}{dx} + f(\bar{u}_0^{(\mp)}, x) = 0, \\ \bar{u}_0^{(\mp)}(i) = u^i, \quad i = 0, 1. \end{cases} \tag{3.5}$$

For $k \geq 1$:

$$\begin{cases} A^{(\mp)}(\bar{u}_0^{(\mp)}, x)\bar{z}_k^{(\mp)}(x) = \dfrac{d\bar{z}_{k-1}^{(\mp)}}{dx} + F_k^{(\mp)}(x), \\ \dfrac{d\bar{u}_k^{(\mp)}}{dx} = \bar{z}_k^{(\mp)}(x), \\ \bar{u}_k^{(\mp)}(0) = 0. \end{cases} \tag{3.6}$$

$$\bar{u}_k^{(\mp)}(x) = \int_0^x \frac{1}{A^{(\mp)}(\bar{u}_0^{(\mp)}, s)}(\bar{u}_{k-1}^{(\mp)}(s) + F^{(\mp)}(s))ds.$$

**Internal layer**

For $(Q_0^{(\mp)}u(\xi), Q_{-1}^{(\mp)}z(\xi))$, we obtain the following problem

$$\begin{cases} \dfrac{dQ_{-1}^{(\mp)}z}{d\xi} = A^{(\mp)}(\varphi_i(x_0) + Q_0^{(\mp)}u, x_0)Q_{-1}^{(\mp)}z, \\ \dfrac{dQ_0^{(\mp)}u}{d\xi} = Q_{-1}^{(\mp)}z, \\ Q_{-1}^{(\mp)}z(\mp\infty) = 0, \quad Q_0^{(\mp)}u(\mp\infty) = 0, \\ Q_{-1}^{(\mp)}z(0) = z_{-1}^{(\mp)}, \quad Q_0^{(\mp)}u(0) = p_0 - \varphi_i(x_0). \end{cases} \tag{3.7}$$

by virtue of Condition 4 it has a solution $Q_0^{(\mp)}u(\xi)$.

# Construction of Asymptotic Solution

For $(Q_k^{(\mp)}u(\xi), Q_{k-1}^{(\mp)}z(\xi))$ with $k \geq 1$, we obtain

$$\begin{cases} \dfrac{dQ_{k-1}^{(\mp)}z}{d\xi} = A^{(\mp)}(\xi)Q_{k-1}^{(\mp)}z + A_u^{(\mp)}(\xi)Q_{-1}^{(\mp)}zQ_k^{(\mp)}u + G_{k-1}^{(\mp)}(\xi), \\ \dfrac{dQ_k^{(\mp)}u}{d\xi} = Q_{k-1}^{(\mp)}z, \\ Q_{k-1}^{(\mp)}z(\mp\infty) = 0, \quad Q_k^{(\mp)}u(\mp\infty) = 0, \\ Q_{k-1}^{(\mp)}z(0) = z_{k-1}^{(\mp)} - \bar{z}_{k-1}^{(\mp)}(x_0), \quad Q_k^{(\mp)}u(0) = p_k - \bar{u}_k(x_0). \end{cases} \tag{3.8}$$

$$\begin{aligned} Q_k^{(\mp)}u(\xi) =& e^{\int_0^\xi A^{(\mp)}(s)ds}\Big(\int_0^\xi \int_{\mp\infty}^s G_{k-1}^{(\mp)}(q)dq e^{-\int_0^s A^{(\mp)}(q)dq} ds\Big) \\ &+ e^{\int_0^\xi A^{(\mp)}(s)ds}(p_k - \bar{u}_k(x_0)). \end{aligned} \tag{3.9}$$

# Construction of Asymptotic Solution

**To find** $p_k$, we use the matching condition for the derivatives at the point $x = x_0$

$$u'^{(-)}(x_0, \varepsilon) = u'^{(+)}(x_0, \varepsilon),$$

taking into account the fact that $u' = z$, we obtain the following equivalent condition

$$z^{(-)}(x_0, \varepsilon) = z^{(+)}(x_0, \varepsilon). \tag{3.10}$$

**First we find** $p_0$. In the approximation in $\varepsilon^{-1}$, the matching condition is written as

$$Q_{-1}^{(-)} z(0) = Q_{-1}^{(+)} z(0).$$

To solve this equation, let

$$H(p) = \int_{\varphi_1(x_0)}^{p} A^{(-)}(s, x_0) ds - \int_{\varphi_2(x_0)}^{p} A^{(+)}(s, x_0) ds. \qquad (3.11)$$

We have verified that $H(p)$ has a unique root $p_0 \in (\varphi_1(x_0), \varphi_2(x_0))$.

**In the sequel, we find $p_k$ with** $k \geq 1$. Now the condition is written as

$$\bar{z}_{k-1}^{(-)}(x_0) + Q_{k-1}^{(-)} z(0) = \bar{z}_{k-1}^{(+)}(x_0) + Q_{k-1}^{(+)} z(0). \tag{3.12}$$

We obtain

$$\begin{aligned} p_k =& (A^{(-)}(p_0, x_0) - A^{(+)}(p_0, x_0))^{-1} [\bar{z}_k^{(+)}(x_0) - \bar{z}_k^{(-)}(x_0) \\ &+ A^{(-)}(p_0, x_0) \bar{u}_k^{(-)}(x_0) - A^{(+)}(p_0, x_0) \bar{u}_k^{(+)}(x_0) \\ &+ \int_{+\infty}^{0} G_{k-1}^{(+)}(s, x_0) ds - \int_{-\infty}^{0} G_{k-1}^{(-)}(s, x_0) ds]. \end{aligned} \tag{3.13}$$

Thus, we construct the coefficient functions of $\bar{u}_k^{(\mp)}(x)$, $Q_k^{(\mp)} u(\xi)$ for any $k$.

# Main results

**Theorem**

*Assume that Conditions 1-4 hold. Then for sufficiently small $\varepsilon > 0$ there exist a smooth solution $u(x,\varepsilon)$ of boundary-value problem* (1.1)*, and this solution can be asymptotically represented as*

$$u(x,\varepsilon) = \begin{cases} \sum_{k=0}^{n+1} \varepsilon^k [\bar{u}_k^{(-)}(x) + Q_k^{(-)} u(\xi)] + O(\varepsilon^{n+2}), & 0 \le x \le x_0, \\ \sum_{k=0}^{n+1} \varepsilon^k [\bar{u}_k^{(+)}(x) + Q_k^{(+)} u(\xi)] + O(\varepsilon^{n+2}), & x_0 \le x \le 1. \end{cases}$$

# Example

We consider the boundary-value problem

$$\varepsilon u'' = \begin{cases} (1-x)u' + 6x(1-x), & 0 < x < \frac{1}{2}, \\ (x-2)u' + 2x(x-2), & \frac{1}{2} < x < 1, \end{cases} \tag{4.1}$$

$$u(0) = 0, \quad u(1) = 1. \tag{4.2}$$

The corresponding degenerate equations have the root $\varphi_1(x) = -3x^2$ for $x \in [0, \frac{1}{2}]$ and the root $\varphi_2(x) = x^2$ for $x \in [\frac{1}{2}, 1]$.
It is easy to verify that all conditions of the Theorem are fulfilled, where $p_0$ is defined by the equation
$H(p) = \int_{-3/4}^{p} \frac{1}{2} ds + \int_{1/4}^{p} \frac{3}{2} ds$, and $p_0 = 0$.

# Example

Thus, Theorem implies that problem (4.1) and (4.2) has a solution $u = u(x, \varepsilon)$ satisfying the condition

$$u(x,\varepsilon) = \begin{cases} -3x^2 + \dfrac{3}{4} e^{\frac{1}{2}\xi} + O(\varepsilon), & x \in [0, 1/2], \\ x^2 - \dfrac{1}{4} e^{-\frac{3}{2}\xi} + O(\varepsilon), & x \in (1/2, 1]. \end{cases} \tag{4.3}$$

# Example

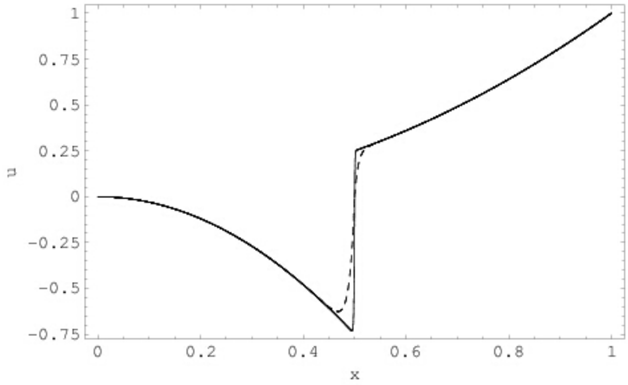


Figure: Asymptotic solution of problem (4.1) and (4.2) with $\varepsilon = 0.01$(dotted) and $\varepsilon = 0.001$(solid).

Thank you!