Least square finite element method for non-Newtonian fluid model

Speaker: Yu Zhou
Supervisor: Lei Hou
Shanghai University

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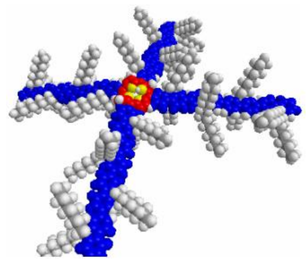
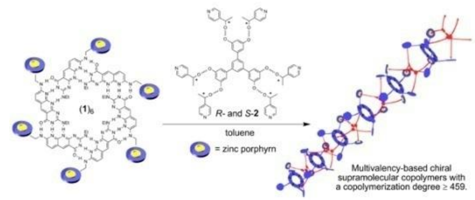
1 Background Rheology



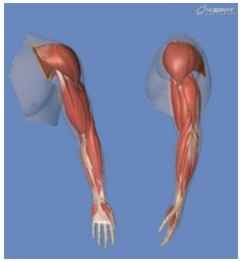


High polymer

Polymer

Mud

Food



Muscle

Blood

BACKGROUND

Non-Newtonian fluid model

Non-Newtonian viscoelastic flows are found in several industrial and biological applications, such as polymer processes, coating and extrusion of polymeric material and artificial organs. Due to the hyperbolic character of the constitutive eqution, numerial simulation of viscoelastic flows is a difficult and expensive task.

BACKGROUND

Non-Newtonian fluid model

Let Ω be a bounded, connected open set in R^2 with Lipschitzian boundary Γ. For the incompressible flows, the continuity and momentum equations are given by:

$$\nabla \cdot u = 0$$

and

$$\mathrm{Re}\frac{Du}{Dt} \equiv \mathrm{Re}(\frac{\partial u}{\partial t} + u \cdot \nabla u) = \nabla \cdot T - \nabla p + f$$

Where Re is the Reynolds number, u the velocity vector, p the pressure, T the extra stress tensor, f the density and D/Dt the substantial derivative.

Non-Newtonian fluid model

The extra stress can be divided into a viscous contribution and a viscoelastic contribution, i.e.

$$T = 2(1-\alpha)D(u) + \tau$$

The viscoelastic stress tensor τ satisfies the following PTT constitutive equation:

$$F(\tau)\tau + \lambda\left(\frac{D\tau}{Dt} - g(\tau, \nabla u)\right) = 2\alpha D(u)$$

Where $F(\tau)$ has the following form:

$$F(\tau) = \exp\left(\frac{\varepsilon\lambda}{\alpha} tr(\tau)\right)$$

Where λ is Weissenberg number and g is a bilinear mapping defined by

$$g(\tau, \nabla u) = \tau\nabla u + \nabla u^T \tau$$

BACKGROUND

Consider the following differential equation:

$$\begin{cases} Lu = f, x \in \Omega, \\ Ru = g, x \in \Gamma, \end{cases}$$

Consider least square functional:

$$J(u; f, g) = \|Lu - f\|_{H_\Omega}^2 + \|Ru - f\|_{H_\Gamma}^2$$

and unconstriant optimization problem:

$$\min_{u \in S} J(u; f, g).$$

The least square finite element method is choosing a finite element subspace $S_h \subset S$, and limting the optimization problem in the subspace. The approximation solutions $u_h \in S_h$ of least square method is the solution of following optimization problem:

$$\min_{u_h \in S_h} J(u; f, g).$$

And we can obtain the Euler-Lagrange equation of the optimization problem:

$$\begin{cases} B(u,v) = F(v), & \forall v \in S \\ B(u_h, v_h) = F(v_h), & \forall v_h \in S_h \end{cases}$$

Where $B(u,v) = (Lu, Lv)_{H_\Omega} + (Ru, Rv)_{H_\Gamma}$ and $F(v) = (Lu, f)_{H_\Omega} + (Ru, g)_{H_\Gamma}$

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PTT fluid model

Consider the following PTT fluid model:

$$
(P)\begin{cases}
-\nabla\cdot T+\nabla p=f & x\in\Omega\\
\nabla\cdot u=0 & x\in\Omega\\
T-2(1-\alpha)D(u)-\tau=0 & x\in\Omega\\
F(\tau)\tau+\lambda(u\cdot\nabla)\tau-\lambda g(\tau,\nabla u)=2\alpha D(u) & x\in\Omega\\
u=0 & x\in\Gamma
\end{cases}
$$

We decouple the system into Stokes equation and constitutive equation. The algorithm as following:
(1)Suppose that τ is obtianed from the previous iteration, we slove the following Stokes system to get u and p by WLS method.

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PTT fluid model

$$(P1)\begin{cases} -\nabla \cdot T + \nabla p = f & x \in \Omega \\ \nabla \cdot u = 0 & x \in \Omega \\ T - 2(1-\alpha)D(u) - \tau = 0 & x \in \Omega \\ u = 0 & x \in \Gamma \end{cases}$$

(2)Using the value of u computed by the first step, we solve this constitutive equation:

$$F(\tau)\tau + \lambda(u \cdot \nabla)\tau - \lambda g(\tau, \nabla u) = 2\alpha D(u) \qquad (P2)$$

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WLS Method

We define the following function spaces for the unkown functions: the velocity u, the pressure p, the extra-stress tensor τ and the viscoelastic stress σ

$$V = H_0^1(\Omega)^d = \{v \in H_0^1(\Omega)^d : v|_\Gamma = 0\}$$

$$Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega adx = 0\}$$

$$T = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji}, \tau_{ij} \in L^2(\Omega)\}$$

$$S = \{\sigma \in T, v \cdot \nabla\sigma \in L^2(\Omega), \forall v \in V\}$$

Let $X = V \times Q \times T \times S$

Then we define the finite element spaces for unkown functions u, p, T as follows:

RESEARCH

$$V_h = \{v \in V \cap C(\bar{\Omega})^2, v|_k \in P_{l_1}(K)^2, \forall K \in \mu^h\}$$

$$Q_h = \{q \in Q \cap C(\bar{\Omega}), q|_k \in P_{l_2}(K), \forall K \in \mu^h\}$$

$$S_h = \{\sigma \in S \cap C(\bar{\Omega})^4, \sigma|_k \in P_{l_3}(K)^4, \forall K \in \mu^h\}$$

Let $X_h = V_h \times Q_h \times S_h$ and we establish the weighted least square functional of problem (P1) as follows:

$$J(v,q,\sigma) = h^{-2}\|\sigma - 2(1-\alpha)D(v) - \tau\|^2 + Lh^{-2}\|\nabla \cdot v\|^2 + \|-\nabla \cdot \sigma + \nabla q - f\|^2 \quad (F1)$$

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Then the least spuare variation problem of (P1) is find the minimizer of (F1) in space X, that is find $(u,p,T)\in X$ satisfies:

$$J(u,p,T)=\inf_{(v,q,\sigma)\in X} J(v,q,\sigma)$$

If (u,p,T) is the minimizer point of the problem above, then (u,p,T) must satisfies the following Euler-Lagrange equation:

$$Q(u,p,T;v,q,\sigma)=F(\tau;v,q,\sigma),\forall(v,q,\sigma)\in X$$

Where

$$Q(u,p,T;v,q,\sigma)=\int_\Omega h^{-2}(T-2(1-\alpha)D(u)):(\sigma-2(1-\alpha)D(v))$$
$$+(-\nabla\cdot T+\nabla p)\cdot(-\nabla\cdot\sigma+\nabla q)+Lh^{-2}(\nabla\cdot u)(\nabla\cdot v)dx$$

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And

$$F(\tau;v,q,\sigma)=\int_{\Omega}h^{-2}\tau:(\sigma-2(1-\alpha)D(v))+f\cdot(-\nabla\cdot\sigma+\nabla q)dx$$

The WLS finite element method for (P1) is find the minimizer of functional (F1) in the space X_h, that is find $(u_h,p_h,T_h)\in X_h$ satisfies:

$$J(u_h,p_h,T_h)=\inf_{(v_h,q_h,\sigma_h)\in X_h}J(v_h,q_h,\sigma_h)$$

And the minimizer (u_h,p_h,T_h) satisfies the following Euler-Lagrange equation as well

$$Q(u_h,p_h,T_h;v_h,q_h,\sigma_h)=F(\tau_h;v_h,q_h,\sigma_h),\forall(v_h,q_h,\sigma_h)\in X_h$$

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Error Estimate

Theorem1 Assume that (u,p,T) is the solution of PTT problem (P1), then WLS finite element method solution (u_h,p_h,T_h) satisfies the following error estimation

$$\|u-u_h\|_1+\|p-p_h\|+\|T-T_h\|\le C\|\tau-\tau_h\|+O(h^2)$$

Theorem2 Assume that the problem (P) has a solution (u,p,T,τ). The finite element solution (u_h,p_h,T_h,τ_h) by the WLS/SUPG method satisfies

$$\|u-u_h\|_1+\|p-p_h\|+\|T-T_h\|+\|\tau-\tau_h\|\le Ch^{3/2}$$

if α and λ are small suffiicently and $\nabla u_h,\tau_h\in L^\infty(\Omega)$

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3 Numerical results

In this section, we present some numerical examples obtained by the weighted least-squares finite element method. We take for the domain the unit square $\Omega=[0,1]\times[0,1]$ with Dirichlet boundary conditions (see Fig. 1)

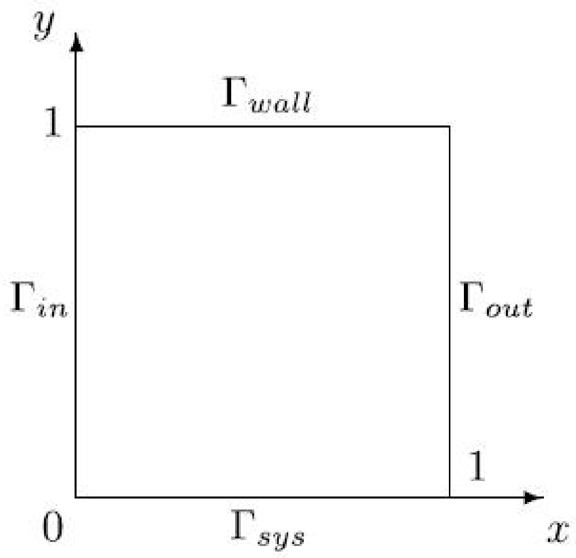

Fig. 1. The geometry and boundry conditions

RESULT

3 Numerical results

We choose the exact solution (u, p, σ) as follows:

$$u = \begin{pmatrix} 1 - y^4 \\ 0 \end{pmatrix} \qquad p = -x^2 \qquad \sigma = \begin{pmatrix} 32\lambda\alpha y^6 & -4\alpha y^3 \\ -4\alpha y^3 & 0 \end{pmatrix}$$

The parameters in the constitutive equation are set as a =1and α =1/9. Therefore, the right-hand side terms of the momentum and constitutive equations are given by

$$f = \begin{pmatrix} 12y^2 - 2x \\ 0 \end{pmatrix} \qquad f_{stress} = \begin{pmatrix} 32\lambda^2 y^6/9 + 1024\varepsilon\lambda^{12}/9 & 128\lambda^2(1-\varepsilon)y^9/9 \\ 128\lambda^2(1-\varepsilon)y^9/9 & -32\lambda y^6/9 \end{pmatrix}$$

RESULT

3 Numerical results

Meshes	m	No. of element	No. of nodes	No. of unknows	h
M1	8	128	81	646	0.1768
M2	16	512	289	2438	0.0884
M3	24	1152	625	5328	0.0589

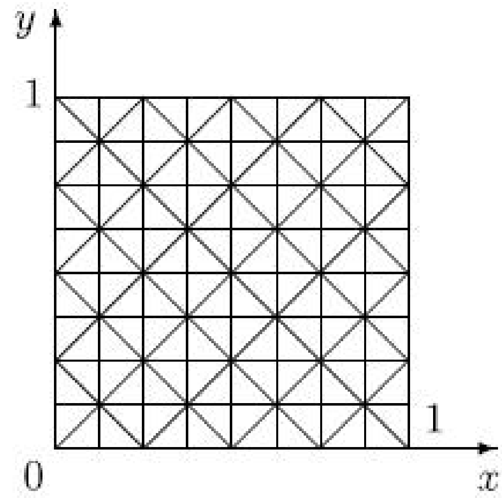

Fig.2. Union Jack grid for m=8

RESULT

3 Numerical results

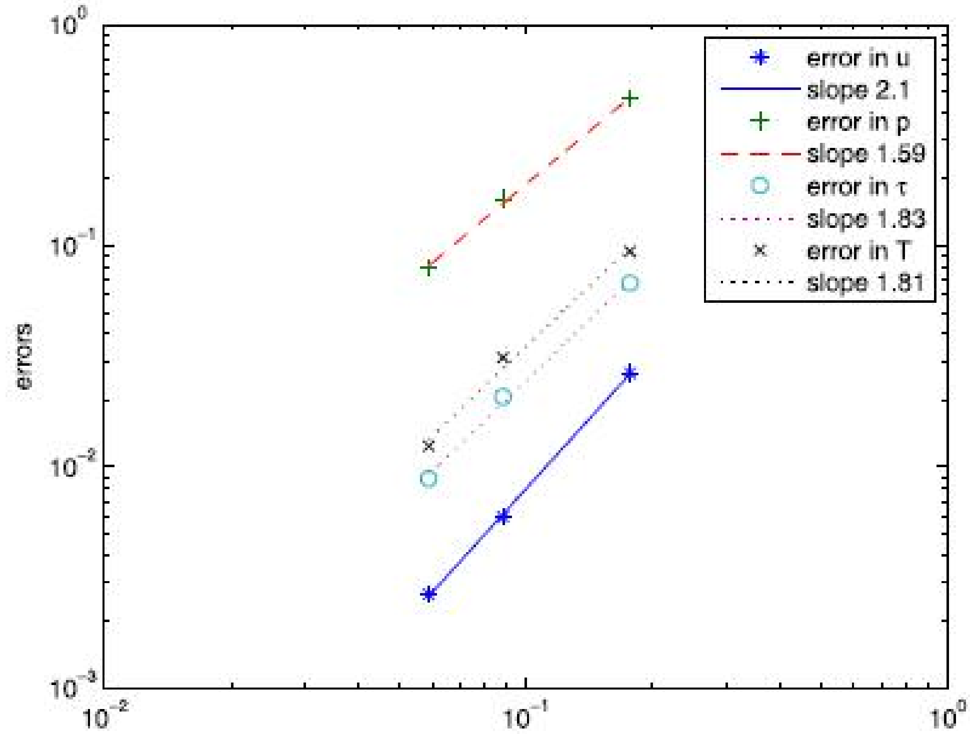

Fig. 3. L^2 errors in u, p, τ and T for Oldroyd-B model with λ=0.5

RESULT

3 Numerical results

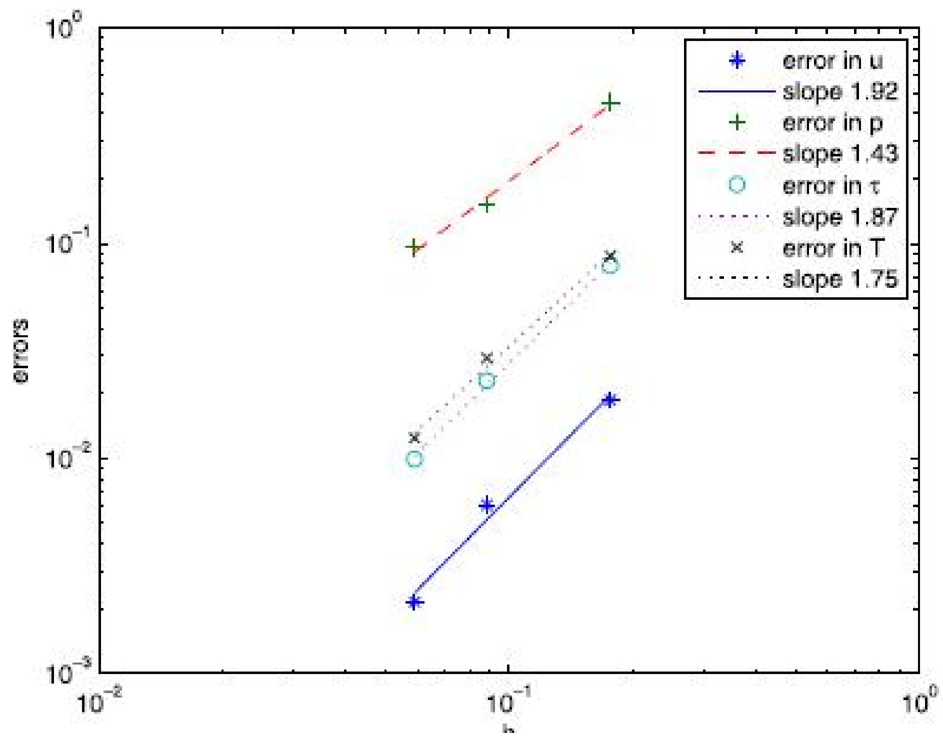

Fig. 3 L^2 errors in u, p, σ and T for Oldroyd-B model with λ=21

RESULT

3 Numerical results

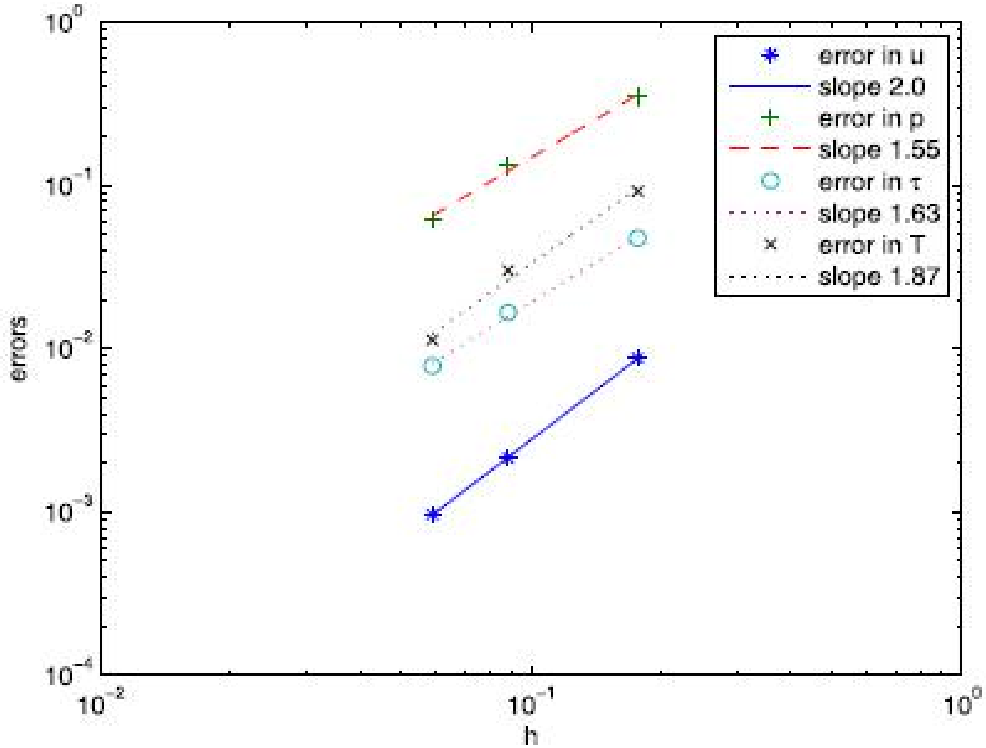

Fig. 4 L^2 errors in u, p, σ and T for PTT model with $\lambda=0.5$

RESULT

4 Conclusions

Weissenberg number λ =0.5. The rates of convergence in L2errors for u, p, τ and T are 2.1, 1.59, 1.83 and 1.81, respectively. The results at λ =21are shown in Fig.3. After that, we consider the PTT viscoelastic model. The parameter $\varepsilon$is set to 0.2. The upper limiting Weissenberg number is quickly reached at 3.5. In Fig.4, the results are obtained by using the three meshes at λ =0.5. In our numerical results, the convergence rates of the finite element solutions are nearly quadratic for velocity and superlinear for stress and pressure.

RESULT

THE END

Thank you!