Renormalization group theory and its application to highly oscillatory problems

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Outline

Introduction & motivations.

Renormalization group (RG) theory

- Chen *et al* (Phys. Rev. Lett. 1994) Renormalization group theory for global asymptotic analysis.
- Chen *et al* (Phys. Rev. E. 1996) Renormalization group and singular perturbations: multiple scales, boundary layers, and reductive perturbation theory.

- Main goal: To compute the effective approximate solution of different kinds of singular perturbation problems in a unified manner.
- Advantage: It starts with a naive expansion and does not require any further a priori assumptions regarding the structure of the perturbation series, like an anticipation of scales involved, as is done in multiple scale analysis

RG theory: recent works

- Ei *et al* (Ann. Physics, 2000) Renormalization-group method for reduction of evolution equations: invariant manifolds and envelopes.
- Ziane (J. Math. Phys 2003) On a certain renormalization group method.
- Goldenfeld *et al* (J. Statis. Phys. 2006) Renormalization Group Approach to Multiscale Modelling in Materials Science.
- Chiba (SIAM J. Appl. Dyn. Syst. 2009) Extension and Unification of Singular Perturbation Methods for ODEs Based on the Renormalization Group Method.
- Kirkinis (SIAM Review. 2012) The Renormalization Group: A Perturbation Method for the Graduate Curriculum.
- . . .

RG theory: two cases

Ziane, 2003, JMP

$$\begin{cases} \dot{\mathbf{x}} + \frac{1}{\varepsilon}A\mathbf{x} = \mathbf{f}(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases}$$

- A: a complex diagonalizable matrix;
- $\mathbf{f}(\mathbf{x})$: a polynomial nonlinear term.

Chiba, 2009, SIAM. JADS

$$\dot{\mathbf{x}} = \varepsilon \mathbf{g}(\mathbf{x}, t, \varepsilon), \ \ \mathbf{x} \in U,$$

- U: an open set in $\mathbb{C}^n$, the closure $\bar{U}$ is compact;
- $\mathbf{g}(\mathbf{x}, t, \varepsilon)$: almost-periodic in t, and the set of corresponding Fourier exponents having no accumulation on $\mathbb{R}$.

Motivation 1

1. How about the theory for more general cases?
2. How about the theory for other particular cases?

Highly oscillatory second order ODEs

$$\begin{cases} \varepsilon^2 \ddot{\mathbf{y}}(t) + (A + \frac{1}{\varepsilon^2} B)\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) = 0, \mathbf{y} \in \mathbb{C}^d, \ t > 0, \\ \mathbf{y}(0) = \eta_1, \quad \dot{\mathbf{y}}(0) = \frac{\eta_2}{\varepsilon^2}, \end{cases} \tag{1}$$

- ε: small real parameter $|\varepsilon| \ll 1$;
- $A(B)$: nonnegative (positive) definite matrix;
- $\mathbf{f}(e^{\mathrm{i}s}\mathbf{y}) = e^{\mathrm{i}s}\mathbf{f}(\mathbf{y}), \forall s \in \mathbb{R}$.

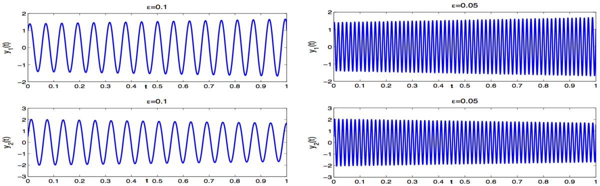
Nonlinear Klein-Gordon equation in the nonrelativistic limit regime

$$\begin{cases} \varepsilon^2 \partial_{tt} u - \triangle u + \frac{1}{\varepsilon^2} u + f(u) = 0, \ \mathbf{x} \in \mathbb{R}^n, \ t > 0, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), \ \partial_t u(\mathbf{x}, 0) = \frac{1}{\varepsilon^2} \gamma(\mathbf{x}), \end{cases} \tag{2}$$

- ε: small real parameter $|\varepsilon| \ll 1$ inversely proportional to the speed of light;
- ϕ, γ: real-valued functions;
- $f(e^{\mathrm{i}s} u) = e^{\mathrm{i}s} f(u), \forall s \in \mathbb{R}$.

High oscillation property

Propagates highly oscillatory waves with wavelength at $O(\varepsilon^2)$ and amplitude at $O(1)$.



- Difficulties in the asymptotic analysis.
- Burdens in practical computation, making the approximation extremely challenging and costly in the regime of $0 < \varepsilon \ll 1$.

Known works

- Machihara *et al.* (Math. Ann. 2002) Nonrelativistic limit in the energy space for non-linear Klein-Gordon equations.
- Masmoudi & Nakanishi (Math. Ann. 2002) From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations.
- Bao & Dong (Numer. Math. 2012) Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime.
- Bao, Dong & Zhao (J. Math. Study. 2014) Uniformly Accurate Multiscale Time Integrators for Highly Oscillatory Second Order Differential Equations.
- $\cdots$

Motivation 2.

1. To make a complete investigation of the asymptotic properties as $\varepsilon \to 0$, from the point of renormalization group theory.
2. To present a rigorous statement of the following assumption proposed to secure the validity of the numerical results:

$$\|\mathbf{y}^{(k)}(t)\| \leq \frac{M}{\varepsilon^{2k}}, \quad k = 0, 1, 2, \cdots \quad x \in [0, T] \subset [0, T^*). \tag{3}$$

RG theory revisited.

KBM condition

$$\dot{\mathbf{x}} = \varepsilon \mathbf{g}(\mathbf{x}, t, \varepsilon), \ \ \mathbf{x} \in U, \tag{4}$$

- U: open set in $\mathbb{C}^n$, and the closure $\bar{U}$ is compact.

($\mathbf{G_1}$): $\mathbf{g}(\mathbf{x}, t, \varepsilon)$ is C^2 in $\mathbf{x} \in U$, C^1 in t and analytic in $\varepsilon \in I_0$, with $I_0 \subset \mathbb{R}$ an open neighborhood of the origin. Furthermore, $\mathbf{g}(\mathbf{x}, t, \varepsilon)$ is Lipschitz continues in $\mathbf{x}$ on U, i.e., there exists a constant $L_U > 0$ such that

$$||\mathbf{g}(\mathbf{x}, t, \varepsilon) - \mathbf{g}(\mathbf{y}, t, \varepsilon)|| \leq L_U ||\mathbf{x} - \mathbf{y}||,$$

for all $\mathbf{x}, \mathbf{y} \in U, t \in \mathbb{R}, \varepsilon \in I_0$.

KBM condition:

$$R(\mathbf{y}, \varepsilon) = \lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \mathbf{g}(\mathbf{y}, s, \varepsilon) ds \tag{5}$$

converges for arbitrary $t_0 \in \mathbb{R}_+$, and is uniform in $\mathbf{y} \in \bar{U}, \varepsilon \in I_0$.

RG step 1: Naive expansion

Expand $\mathbf{x}$ as $\mathbf{x} = \mathbf{x}^{(0)} + \varepsilon \mathbf{x}^{(1)} + \varepsilon^2 \mathbf{x}^{(2)} + \cdots$,

$$\dot{\mathbf{x}}^{(0)} = 0, \tag{6}$$

$$\dot{\mathbf{x}}^{(1)} = G_1(\mathbf{x}^{(0)}, t), \tag{7}$$

$$\dot{\mathbf{x}}^{(2)} = G_2(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, t), \tag{8}$$

$$\cdots,$$

with

$$\begin{aligned} G_1(\mathbf{x}^{(0)}, t) &= \mathbf{g}(\mathbf{x}^{(0)}, t, 0), \\ G_2(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, t) &= \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}^{(0)}, t, 0)\mathbf{x}^{(1)} + \frac{\partial \mathbf{g}}{\partial \varepsilon}(\mathbf{x}^{(0)}, t, 0), \\ \cdots & \quad . \end{aligned}$$

RG step 1: Naive expansion

$$
\begin{aligned}
\mathbf{x}^{(0)} &= \xi_0 \\
\mathbf{x}^{(1)} &= \int_0^t G_1(\xi_0, s)ds = R_1(\xi_0)t + N_1(\xi_0, t), \\
\mathbf{x}^{(2)} &= \int_0^t G_2(\xi_0, R_1(\xi_0)s + N_1(\xi_0, s), s)ds \\
&= (R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0))t + N_2(\xi_0, t) + \frac{1}{2}\frac{\partial R_1}{\partial \xi_0}(\xi_0)R_1(\xi_0))t^2, \\
&\vdots
\end{aligned}
$$

with

$$
\begin{aligned}
R_1(\xi_0) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t G_1(\xi_0, s)ds, \\
N_1(\xi_0, t) &= \int_0^t (G_1(\xi_0, s) - R_1(\xi_0))ds;
\end{aligned}
$$

RG step 1: Naive expansion

$$
\begin{aligned}
R_2(\xi_0) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t (G_2(\xi_0, N_1(\xi_0, s), s) - \frac{\partial N_1}{\partial \xi_0}(\xi_0, s) R_1(\xi_0)) ds, \\
N_2(\xi_0, t) &= \int_0^t (G_2(\xi_0, N_1(\xi_0, s), s) - \frac{\partial N_1}{\partial \xi_0}(\xi_0, s) R_1(\xi_0) - R_2(\xi_0)) ds, \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}(\xi_0, t, \varepsilon) &= \xi_0 + \varepsilon(R_1(\xi_0)t + N_1(\xi_0, t)) + \varepsilon^2((R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t) R_1(\xi_0))t \\
&\quad + N_2(\xi_0, t) + \frac{1}{2}\frac{\partial R_1}{\partial \xi_0}(\xi_0) R_1(\xi_0))t^2) + O(\varepsilon^3). \qquad (9)
\end{aligned}
$$

RG step 2: Renormalization

Propose an ansatz free parameter $\mu \in \mathbb{R}$ into $\mathbf{x}(\xi_0, t, \varepsilon)$ to replace t^i by $t^i - \mu^i + \mu^i, i = 1, 2, \cdots$.

$$\begin{aligned}
\mathbf{x}(\xi_0, t, \varepsilon) \quad = \quad & \xi_0 + \varepsilon(R_1(\xi_0)(t - \mu + \mu) + N_1(\xi_0, t)) + \\
& \varepsilon^2((R_2(\xi_0) + \frac{\partial N_1}{\partial \xi_0}(\xi_0, t)R_1(\xi_0))(t - \mu + \mu) + \\
& N_2(\xi_0, t) + \frac{1}{2}\frac{\partial R_1}{\partial \xi_0}(\xi_0)R_1(\xi_0))(t^2 - \mu^2 + \mu^2))) + \cdots .
\end{aligned}$$

RG step 2: Renormalization

Construct

$$\xi(\mu,\varepsilon)=\xi_0+\varepsilon A_1(\mu)+\varepsilon^2 A_2(\mu)+\cdots$$

such that

$$\mathbf{x}(\xi_0,t,\varepsilon)=\tilde{\mathbf{x}}(\xi(\mu,\varepsilon),t-\mu,t,\varepsilon), \tag{10}$$

without the secular terms and the ones as $\varepsilon R_1(\xi)\mu, \varepsilon^2((R_2(\xi)+\frac{\partial N_1}{\partial \xi}(\xi,t)R_1(\xi))\mu,\cdots$.

RG step 2: Renormalization

$$\xi_0 = \xi(\mu,\varepsilon) - \varepsilon R_1(\xi(\mu,\varepsilon))\mu - \varepsilon^2 (R_2(\xi(\mu,\varepsilon))\mu + \frac{\partial R_1}{\partial \xi}(\xi(\mu,\varepsilon))R_1(\xi(\mu,\varepsilon))\mu^2) + \cdots ,$$

$$\begin{aligned}
\tilde{\mathbf{x}}(\xi(\mu,\varepsilon), t-\mu, t, \varepsilon) &= \mathbf{x}(\xi_0, t, \varepsilon) \\
&= \xi(\mu,\varepsilon) + \varepsilon(R_1(\xi(\mu,\varepsilon))(t-\mu) + N_1(\xi(\mu,\varepsilon),t)) + \\
&\quad \varepsilon^2((R_2(\xi(\mu,\varepsilon)) + \frac{\partial N_1}{\partial \xi}(\xi(\mu,\varepsilon),t)R_1(\xi(\mu,\varepsilon)))(t-\mu) \\
&\quad + N_2(\xi(\mu,\varepsilon),t) + \frac{1}{2}\frac{\partial R_1}{\partial \xi}(\xi(\mu,\varepsilon))R_1(\xi(\mu,\varepsilon)))(t-\mu)^2)) + \cdots .
\end{aligned}$$

RG step 2: Renormalization

$$\frac{\partial}{\partial \mu}\tilde{\mathbf{x}}(\xi(\mu,\varepsilon), t-\mu, t, \varepsilon) = \frac{\partial}{\partial \mu}\mathbf{x}(\xi_0, t, \varepsilon) \equiv 0.$$

Especially, by taking $\mu = t$, we get the renormalized equation:

$$\dot{\xi} = \varepsilon R_1(\xi) + \varepsilon^2 R_2(\xi) + \cdots . \tag{11}$$

with the initial value condition $\xi(0) = \xi_0$, and the corresponding approximate solution:

$$\mathbf{x}(\xi_0, t, \varepsilon) = \xi(t) + \varepsilon N_1(\xi(t), t)) + \varepsilon^2 N_2(\xi(t), t) + \cdots .$$

RG step 2: Asymptotic results

(First order) RG equation

$$\dot{\xi} = \varepsilon R_1(\xi). \tag{12}$$

(First order) approximate solution

$$\hat{\mathbf{x}}(t) = \xi(t) + \varepsilon N_1(\xi(t), t).$$

RG step 3: Estimation

By $\tau = \varepsilon t$, (12) becomes

$$\frac{d\eta}{d\tau} = R_1(\eta). \tag{13}$$

Theorem 1.

Assume that $\mathbf{g}(\mathbf{x}, t, \varepsilon)$ satisfies (G_1) and the KBM condition. Let $\mathbf{x}(t)$ be a solution of (4), $\eta(\tau)$ be the solution of (13) with $\eta(0) = \mathbf{x}(0)$ and the maximum existence interval (a, b). Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\| \mathbf{x}(t) - \hat{\mathbf{x}}(t) \| < C\delta(\varepsilon),$$

as long as $\frac{T_1}{\varepsilon} < t < \frac{T_2}{\varepsilon}$, where C is a positive constant independent to ε, and $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

Periodic case

Corollary 1.

Assume that $\mathbf{g}(\mathbf{x}, t, \varepsilon)$ satisfies (G_1), and it is periodic in t. Let $\mathbf{x}(t)$ be a solution of (4), $\eta(\tau)$ be the solution of (13) with $\eta(0) = \mathbf{x}(0)$ and the maximum existence interval (a, b). Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\| \mathbf{x}(t) - \hat{\mathbf{x}}(t) \| < C\varepsilon, \tag{14}$$

as long as $\frac{T_1}{\varepsilon} < t < \frac{T_2}{\varepsilon}$, where C is a positive constant independent to ε.

- A same result can be found in [Chiba. SIAM J. Appl. Dyn. Syst. 2009].

Quasi-periodic case

(G_2): $\mathbf{g}(\mathbf{x}, t, \varepsilon)$ is sufficiently smooth in $(\mathbf{x}, t, \varepsilon)$ and quasi-periodic in t, i.e., it can be rewritten as the following form

$$\mathbf{g}(\mathbf{x}, t, \varepsilon) = \sum_{\nu \in \mathbb{Z}^d} \mathbf{g}_\nu(\mathbf{x}, \varepsilon) e^{\mathrm{i}\langle \omega, \nu \rangle t} = \sum_{\nu \in \mathbb{Z}^d} \sum_{l=0}^{\infty} \varepsilon^l \mathbf{g}_{\nu l}(\mathbf{x}) e^{\mathrm{i}\langle \omega, \nu \rangle t}, \tag{15}$$

where $\omega \in \mathbb{R}^d$ is a constant vector, $\mathrm{i} = \sqrt{-1}$, and

$$\max\{\|\mathbf{g}_{\nu l}(\mathbf{x})\|, \|\nabla \mathbf{g}_{\nu l}(\mathbf{x})\|\} \leq \Gamma_0 \Gamma_1^l e^{-\sigma|\nu|} \tag{16}$$

is valid for some $\Gamma_0, \Gamma_1, \sigma > 0$ and any $l \in \mathbb{N}, \mathbf{x} \in U$.

Diophantine conditions

Bryuno-Diophantine condition

For fixed vector $\eta \in \mathbb{R}^l$, let

$$\alpha_m^l(\eta) = inf\{|\langle \eta, \nu \rangle| | \nu \in \mathbb{Z}^l, \text{such that } 0 < |\nu| < 2^m\}, m \in \mathbb{N}.$$

Then the Bryuno function

$$\mathfrak{B}(\eta) = \sum_{m=0}^{\infty} \frac{1}{2^m} \ln \frac{1}{\alpha_m^l(\eta)} < \infty.$$

GBD condition

For fixed vector $\eta \in \mathbb{R}^l$, decompose η as $\eta = (a_1\eta^{(1)}, \cdots, a_j\eta^{(j)}) \in \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_j}$ with $\eta^i \in \mathbb{Q}^{l_i}, l_i \geq 0, i = 1, \cdots, j,$ $l_1 + \cdots + l_j = l$, such that **a** $= (a_1, \cdots, a_j)$ satisfies the Bryuno - Diophantine condition.

Quasi-periodic case

Theorem 2.

Assume the assumption (G_2) holds, and the corresponding ω satisfies the GBD condition. Let $\mathbf{x}(t)$ be a solution of (4), $\eta(\tau)$ be the solution of (13) with $\eta(0) = \mathbf{x}(0)$ and the maximum existence interval (a, b). Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\| \mathbf{x}(t) - \hat{\mathbf{x}}(t) \| < C\varepsilon, \tag{17}$$

as long as $\frac{T_1}{\varepsilon} < t < \frac{T_2}{\varepsilon}$, where C is a positive constant independent to ε.

Highly oscillatory problems.

Model

$$\begin{cases} \varepsilon^2 \ddot{\mathbf{y}}(t) + (A + \frac{1}{\varepsilon^2} B)\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) = 0, \mathbf{y} \in \mathbb{C}^d, \ t > 0, \\ \mathbf{y}(0) = \eta_1, \quad \dot{\mathbf{y}}(0) = \frac{\eta_2}{\varepsilon^2}, \end{cases} \tag{18}$$

- ε: small real parameter $|\varepsilon| \ll 1$;
- $A(B)$: nonnegative (positive) definite matrix;
- $\mathbf{f}(e^{\mathrm{i}s}\mathbf{y}) = e^{\mathrm{i}s}\mathbf{f}(\mathbf{y}), \forall s \in \mathbb{R}$.

- Without loss of generality, we assume $B = \Lambda^2$ with $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_d\}, \lambda_j > 0, j = 1, \cdots, d$..

Transformed problems

$$\text{By } \epsilon = \varepsilon^2, \quad t = \epsilon x, \quad \mathbf{y}_1(x) = \mathbf{y}(t), \quad \mathbf{y}_2(x) = \mathbf{y}_1'(x), \tag{19}$$

where $'$ denotes the derivation with respect to x.

$$\begin{cases} \begin{pmatrix} \mathbf{y}_1' \\ \mathbf{y}_2' \end{pmatrix} = \begin{pmatrix} O & E \\ -B & O \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} + \epsilon \begin{pmatrix} O \\ -A\mathbf{y}_1 - \mathbf{f}(\mathbf{y}_1) \end{pmatrix}, \\ \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{cases} \tag{20}$$

Transformed problems

By
$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} E & \mathrm{i}\Lambda \\ \mathrm{i}E & \Lambda \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\Lambda x} & O \\ O & e^{-\mathrm{i}\Lambda x} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}. \tag{21}$$

Then (20) becomes

$$\begin{cases} \mathbf{Z}' = \epsilon \mathbf{g}(\mathbf{Z}, x), \\ \mathbf{Z}(0) = \begin{pmatrix} \frac{\eta_1 - \mathrm{i}\Lambda^{-1}\eta_2}{2} \\ \frac{-\mathrm{i}\eta_1 + \Lambda^{-1}\eta_2}{2} \end{pmatrix}. \end{cases} \tag{22}$$

$$\mathbf{g}(\mathbf{Z}, x) = \frac{1}{2} \begin{pmatrix} A(\mathrm{i}\Lambda^{-1}\mathbf{z}_1 - e^{-2\mathrm{i}\Lambda x}\mathbf{z}_2) + \mathrm{i}\Lambda^{-1}e^{-\mathrm{i}\Lambda x}\mathbf{f}(e^{\mathrm{i}\Lambda x}\mathbf{z}_1 + \mathrm{i}\Lambda e^{-\mathrm{i}\Lambda x}\mathbf{z}_2) \\ -A(\Lambda^{-1}e^{2\mathrm{i}\Lambda x}\mathbf{z}_1 + \mathrm{i}\mathbf{z}_2) - \Lambda^{-1}e^{\mathrm{i}\Lambda x}\mathbf{f}(e^{\mathrm{i}\Lambda x}\mathbf{z}_1 + \mathrm{i}\Lambda e^{-\mathrm{i}\Lambda x}\mathbf{z}_2) \end{pmatrix}$$

Renormalization

- Now, assume that $(\lambda_1, \cdots, \lambda_d)$ satisfies the GBD condition.

Approximate solution

$$\hat{\mathbf{Z}}(x) = \mathbf{W}(x) + \epsilon N_1(\mathbf{W}(x), x) := \Phi_1(\mathbf{W}(x), x),$$

with $N_1(\mathbf{W}, x) = \int_0^t (\mathbf{g}(\mathbf{W}, s) - R_1(\mathbf{W}))ds$, and $\mathbf{W}(x) = (\mathbf{w}_1^T(x), \mathbf{w}_2^T(x))^T$ is the solution of the RG equation:

$$\begin{cases} \mathbf{W}' = \epsilon R_1(\mathbf{W}), \\ \mathbf{W}(0) = \begin{pmatrix} \frac{\eta_1 - \mathrm{i}\Lambda^{-1}\eta_2}{2} \\ \frac{-\mathrm{i}\eta_1 + \Lambda^{-1}\eta_2}{2} \end{pmatrix}, \end{cases} \tag{23}$$

$$R_1(\mathbf{W}) = \frac{\mathrm{i}A}{2}\begin{pmatrix} \Lambda^{-1}\mathbf{w}_1 \\ -\mathbf{w}_2 \end{pmatrix} + \frac{1}{2}\lim_{x\to\infty}\frac{1}{x}\int_0^x \begin{pmatrix} \mathrm{i}\Lambda^{-1}e^{-\mathrm{i}\Lambda s}\mathbf{f}(e^{\mathrm{i}\Lambda s}\mathbf{w}_1 + \mathrm{i}\Lambda e^{-\mathrm{i}\Lambda s}\mathbf{w}_2) \\ -\Lambda^{-1}e^{\mathrm{i}\Lambda s}\mathbf{f}(e^{\mathrm{i}\Lambda s}\mathbf{w}_1 + \mathrm{i}\Lambda e^{-\mathrm{i}\Lambda s}\mathbf{w}_2) \end{pmatrix} ds.$$

Approximate results

$$\begin{pmatrix} \hat{\mathbf{y}}_1(t) \\ \hat{\mathbf{y}}_2(t) \end{pmatrix} = \begin{pmatrix} E & \mathrm{i}\Lambda \\ \mathrm{i}E & \Lambda \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\Lambda t/\varepsilon^2} & O \\ O & e^{-\mathrm{i}\Lambda t/\varepsilon^2} \end{pmatrix} \Phi_1(\mathbf{V}(t), t/\varepsilon^2),$$

where

$$\begin{cases} \dot{\mathbf{V}} = R_1(\mathbf{V}), \\ \mathbf{V}(0) = \begin{pmatrix} \frac{\eta_1 - \mathrm{i}\Lambda^{-1}\eta_2}{2} \\ \frac{-\mathrm{i}\eta_1 + \Lambda^{-1}\eta_2}{2} \end{pmatrix}. \end{cases} \tag{24}$$

with maximum existence interval is (a, b) with $a < b \in \mathbb{R} \cup \{\pm\infty\}$.

Approximate results

Theorem 3.

Assume vector $(\lambda_1, \cdots, \lambda_d)$ satisfies the GBD condition. Let $\mathbf{y}(t)$ be the solution of (18), and $\mathbf{V}(t)$ be the solution of (24). Then, for any closed interval $[T_1, T_2] \subset (a, b)$, there exists a constant $\varepsilon_0 > 0$, such that, for any $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \|\mathbf{y}(t) - \hat{\mathbf{y}}_1(t)\| &< C\varepsilon^2, && (25) \\ \|\dot{\mathbf{y}}(t) - \frac{\hat{\mathbf{y}}_2(t)}{\varepsilon^2}\| &< C, && (26) \\ \|\frac{d^k}{dt^k}\mathbf{y}(t)\| &\leq \frac{M}{\varepsilon^{2k}}, && (27) \end{aligned}$$

as long as and $T_1 \leq t \leq T_2$, where C, M are positive constants independent to ε.

Original problems

Corollary 2.

Assume that $B = E$, A is a semi-positive definite matrix, $\eta_1, \eta_2 \in \mathbb{R}^d$, and $\mathbf{f}(\mathbf{y}) = \frac{\partial}{\partial \bar{\mathbf{y}}} h(|\mathbf{y}|^2)$ with $h(r)$ is a polynomial function in $\mathbb{R}$. Let $\mathbf{y}(t)$ be the solution of the singular perturbed problem (18), $\mathbf{V}(t)$ be the solution of (24). Then, for any $T > 0$, there exists a constant $\varepsilon_0 > 0$, such that, for any $k \in \mathbb{N}, 0 < \varepsilon < \varepsilon_0$ and $-T \le t \le T$,

$$\begin{aligned} |\mathbf{y}(t) - \hat{\mathbf{y}}_1(t)| &< C\varepsilon^2, \\ |\dot{\mathbf{y}}(t) - \frac{\hat{\mathbf{y}}_2(t)}{\varepsilon^2}| &< M, \\ |\mathbf{y}^{(k)}(t)| &\le \frac{M}{\varepsilon^{2k}}, \end{aligned}$$

where M is a positive constant independent to ε.

More discussions

- For $d = 1$, by Corollary 2. we have, in fact, obtained the positive answer to the assumption (3) proposed in [Bao *et al* , Numer. Math. 2012; J. Math. Study. 2014].
- The equation (24) is equivalent to

$$\begin{cases} \mathtt{i}\dot{\mathbf{v}}_1 + \frac{1}{2}A\mathbf{v}_1 + \frac{1}{2}\sum_{k=1}^{N}(\sum_{l=1}^{k} C_k^l C_{k-1}^{l-1})kh_k|\mathbf{v}_1|^{2k-2}\mathbf{v}_1 = 0, \\ \mathbf{v}_1(0) = \frac{\eta_1 - \mathtt{i}\eta_2}{2}, \end{cases}$$

which is, in fact, a complex Hamiltonian system.

Other problems

$$\begin{cases} \ddot{\mathbf{y}}(t) + \frac{1}{\varepsilon^2} A\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) = 0, \mathbf{y} \in \mathbb{C}^d, \ t > 0, \\ \mathbf{y}(0) = \eta_1, \ \ \dot{\mathbf{y}}(0) = \frac{\eta_2}{\varepsilon}, \end{cases} \tag{28}$$

$$\begin{cases} \mathrm{i}\dot{\mathbf{y}}(t) + \frac{1}{\varepsilon} A\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) = 0, \mathbf{y} \in \mathbb{C}^d, \ t > 0, \\ \mathbf{y}(0) = \eta_1, \end{cases} \tag{29}$$

- ε: small real parameter $|\varepsilon| \ll 1$;
- A: nonnegative (positive) definite matrix;

Conclusions.

1. Gave a systematic introduction to the singular perturbation renormalization group theory under more general conditions.
2. Presented a new sufficient condition for quasi-periodic cases to the singular perturbation RG theory.
3. Obtained a complete discussion to the highly oscillatory problem (18), and positive answer to the assumption (3) proposed in [Bao *et al* , Numer. Math. 2012; J. Math. Study. 2014].

Thank you!

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