SLOW-FAST SYSTEMS WITH SEVERAL BREAKING PARAMETERS

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Based on articles written in collaboration with

Freddy Dumortier, University of Hasselt, Belgium.

[1]*Multiple Canard Cycles in Generalized Liénard Equations*, J. Diff. Eq 174, (2001).1–29

[2] *Bifurcations of relaxation oscillations in dimension two*, Disc. Cont. Dyn. Sys. Vol. 19, n$^\circ 4$ (2007) 631–674

[3]*Canard cycles with two breaking parameters*, Discrete Contin. Dyn. Sys. 17, n$^\circ 4$, (2007), 787–806

[4]*Multi-Layer canard cycles and translated power functions*, Jour. Diff. Eq. , 244, (2008), 1329–1358

Lilia Mahmoudi, University of Annaba , Algeria.

[5]*Canard cycles of finite codimension with two breaking parameters* Qual. Th. Dyn Syst. 11, no. 1, 167–198 (2012).

Magdalena Caubergh, Universidade Autonoma de Barcelona, Spain.

[6]*Canard cycles with three breaking mechanisms*, Volume in honour of Christiane Rousseau, (2015)

Plan

1. Preliminaries
2. The case $k = 1$
3. The case $k = 2$
4. The general case $k \geq 3$
5. What about $M(k)$?
6. The proof that $M(3) = 5$
7. Open questions

Preliminaries

We consider slow-fast systems defined on a surface M and locally of the form

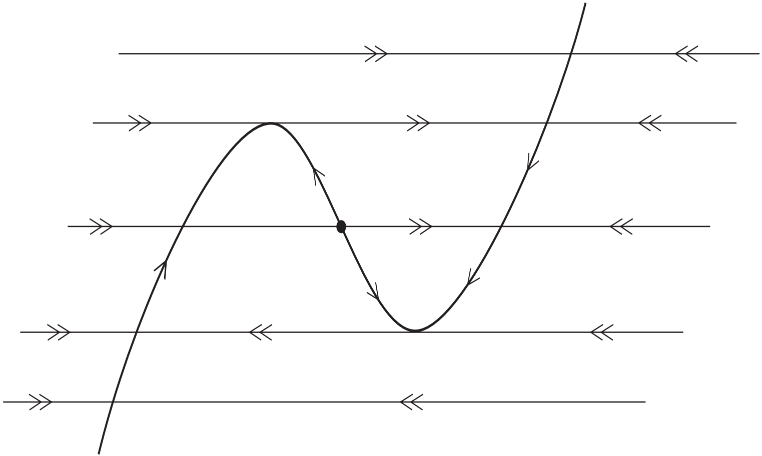
$$X_{\lambda,\varepsilon} : \begin{cases} \dot{x} = f(x, y, \lambda, \varepsilon) \\ \dot{y} = \varepsilon g(x, y, \lambda, \varepsilon), \end{cases}$$

where f, g are smooth functions and (ε, λ) is the parameter, with $\varepsilon \geq 0$ and small.

The **critical set** L_λ locally defined by $\{f(x,y,\lambda,0)=0\}$ is supposed to be a regular curve.

$C_\lambda=\{\frac{\partial f}{\partial x}=0\}$ is the set of **contact points.** Here we will assume that any contact point is quadratic.

A quadratic contact point is a **jump point** (if $g\neq 0$ at this point) or a **turning point** (if $g=0$ at this point).



$L_\lambda \setminus C_\lambda$, is an union of **(normally) hyperbolic** arcs, attracting if $\frac{\partial f}{\partial x} > 0$, repulsing if not.

Slow dynamics on each hyperbolic arc γ is given by : $\dot{y} = g(x_\gamma(y, \lambda), \lambda)$, where $x_\gamma(y, \lambda)$ is implicitely given by $f(x_\gamma(y, \lambda), \lambda) = 0$ along γ. **Fast dynamics** is defined on $M \setminus L_\lambda$.

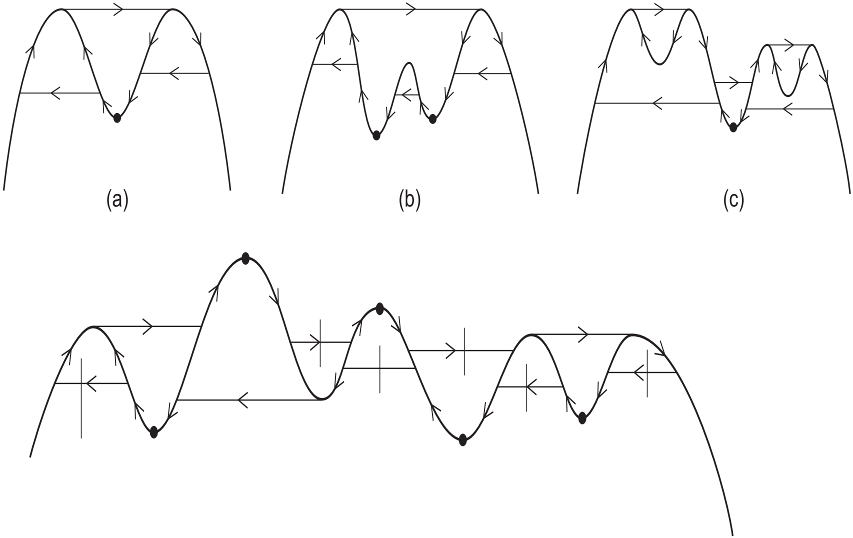
Slow-fast cycle : For a given λ, a simple curve, union of a finite number of arcs contained in $L_\lambda \setminus C_\lambda$, of contact points and fast orbits.

In great generality (for instance in Liénard systems),**limit cycles bifurcate from a singular contact point or from a slow-fast cycle :** they are the limit periodic sets of the system.

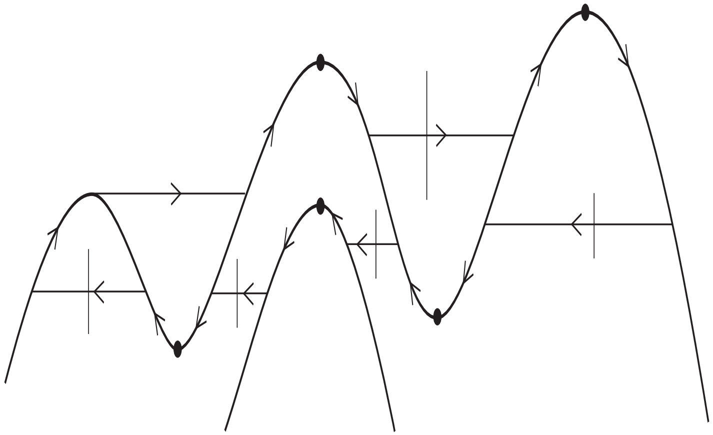
A slow-fast cycle may be :

-common type if all arcs of the slow curve that it contains have the same nature (all attracting or all repulsing),

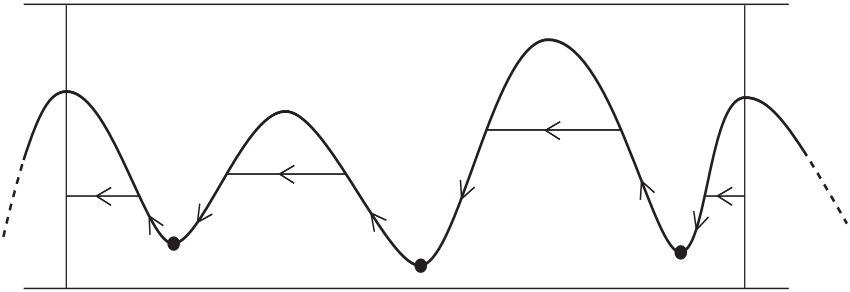
-canard type if it contains at least an attracting and a repulsing arc of the slow curve.



Canard cycles in Liénard systems

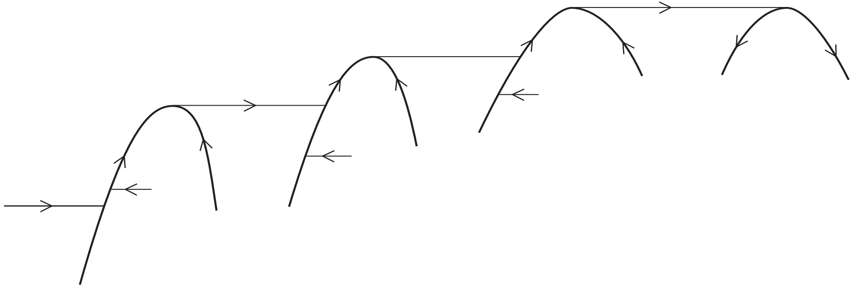


Canard cycles on a non connected critical curve



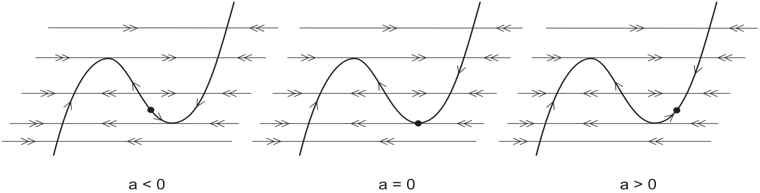
Periodic canard cycle : $M = S^1 \times [0,1]$

A canard cycle is made by a finite number $k \geq 1$ of **attracting sequences** (of regular arcs on L_λ) which altern with the same number of **repulsing sequences.** A repulsing sequence of $X_{\lambda,0}$ is an attracting sequence of $-X_{\lambda,0}$.

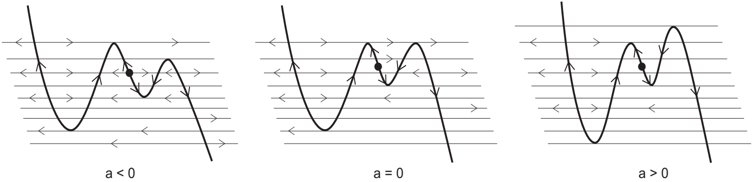


An attracting sequence

One passes from an attracting sequence to the following repulsive sequence by a **canard transition**.



Hopf transition



Jump transition

a is the **breaking parameter** of the canard transition.

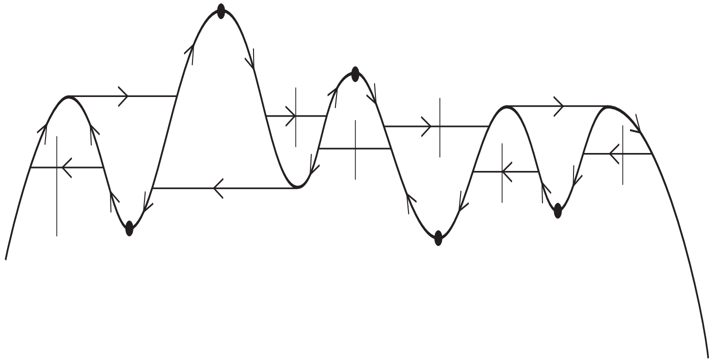
A priori, a is function of the parameter λ. **By genericity we will assume that a is just a component of the parameter λ.**

One returns from a repulsing sequence to the following attracting sequence through a **layer** of fast orbits.

Taking a tranverse section, one parametrizes the position of each fast orbit of a layer by its **layer parameter**.

There is the **same number** k of layers, layer parameters, attracting sequences, repulsing sequences, breaking parameters and canards mechanisms.

Then, we speak indifferently of a **k-layer canard cycle** or of a **canard cycle with k breaking parameters.** We will assume that $\lambda = (a, \nu)$ where $a = (a_1, \ldots, a_k)$ is the vector of the k breaking parameters.



Canard cycle with 6 breaking parameters

A k-layer canard cycle belongs to a k-parameter family of similar canard cycles.

Each canard cycle is associated to $u = (u_1, \ldots, u_k)$, a vector of layer parameters and also to the parameter ν (to have the canard cycle we need that $a = 0$).

Canard cycle for (u, ν) **:** $\Gamma_\nu(u)$.

Along any arc $[a_\lambda, b_\lambda]$ with interior in $L_\lambda \setminus C_\lambda$, we can define the **slow divergence integral** :

$$\mathrm{Int}([a_\lambda, b_\lambda], \lambda) = \int_{[a_\lambda, b_\lambda]} Div(X_{\lambda,0})ds,$$

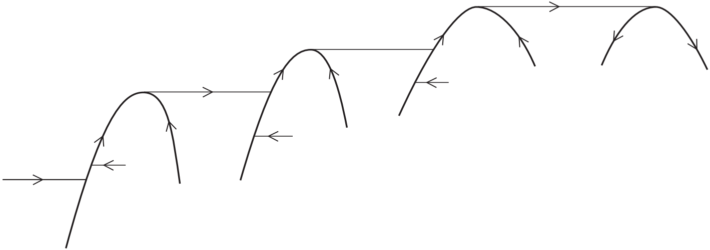
where s is the **slow time.** As the contact points are quadratic, this integral is well-defined, even if a_λ or b_λ is a contact point.

For instance, for the Liénard sytem $\dot{x} = y - F(x,\lambda); \quad \dot{y} = \varepsilon g(x,\lambda)$, we have

$$\mathrm{Int}(\gamma_\lambda) = -\int_{x_1}^{x_2} \frac{1}{g(x,\lambda)}\Big(\frac{\partial F}{\partial x}(x,\lambda)\Big)^2 dx,$$

for the arc γ_λ of graph : $[x_1, x_2] \to (x, F(x,\lambda))$.

We consider now an attracting sequence A_λ. beginning at the ω-limit point of $u \in \Sigma$, a starting section and finishing at a jump point p_λ.



The **slow divergence integral $\mathrm{Int}(A_\lambda)$ of this attracting sequence** is the sum of of the slow divergence integrals along the hyperbolic attracting arcs of A_λ. It is a smooth function of (u, λ). We will write it, for instance :

$$\mathrm{Int}(A_\lambda) = I(u, \lambda) < 0.$$

Let T be a ending section, cutting transversally in its interior the fast orbit with p_λ as α-limit point. The structure of the **transition map** $T(u, \lambda, \varepsilon)$ from Σ to T, along the trajectories of $X_{\lambda,\varepsilon}$ is given by the following

Theorem

There exist functions $\tilde{I}(u, \lambda, \varepsilon)$ and $\varphi(\lambda, \varepsilon)$ such that

$$T(u, \lambda, \varepsilon) = e^{\frac{\tilde{I}(u,\lambda,\varepsilon)}{\varepsilon}} + \varphi(\lambda, \varepsilon)$$

such that $\tilde{I}(u, \lambda, 0) = I(u, \lambda)$ (the slow divergence integral of A_λ) and $\tilde{I}, \varphi$ are **smooth in** $(u, \lambda, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon)$.

We have also to consider the case of an attracting sequence ending at a turning point p_λ, called a **Hopf attracting sequence**. In this case, we take an ending section T after a blowing up of the turning point.

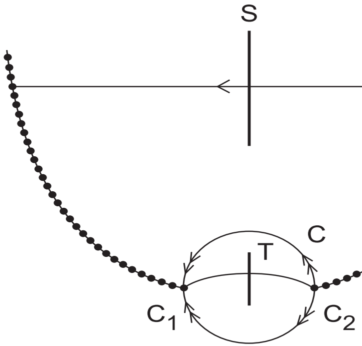


Figure 2: Blow up of turning point

We have a similar formula as above for the transition map $T(u, \lambda, \varepsilon)$. The only difference is that functions $\tilde{I}, \varphi$ are now **smooth in** $(u, \lambda, \varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon)$, and then remain ε-**regularly smooth in** (u, λ).

We will just retain that $\tilde{I}, \varphi$ are ε-**regularly smooth in** (u, λ). This means that these functions are smooth in (u, λ) and that any of **their partial derivatives in** (u, λ) **are continuous in** ε.

We consider a k-layer canard cycle family $\Gamma_\nu(u)$, taking a section Σ_i for each fast layer (parametrized by the variable u_i) and a section T_i at each canard mechanism (T_i following Σ_i).

At each attracting sequence of $\Gamma_\nu(u)$ is associated a slow divergence integral $I_{i,i}(u,\nu)$ and at each repulsing sequence of $\Gamma_\nu(u)$ the slow divergence integral $I_{i,i-1}(u,\nu)$ for the corresponding attracting sequence of $-X_{\lambda,\varepsilon}$. The indice i is taken in $\mathbb{Z}/k\mathbb{Z}$. With this choice, all the integrals $I_{i,j}$ are strictly negative.

In [2] we have obtained the following system of k equations for the bifurcating limit cycles, for $i = 1, \cdots, k$:

$$\exp \frac{\tilde{I}_{i,i-1}(u_{i-1}, a, \nu, \varepsilon)}{\varepsilon} - \exp \frac{\tilde{I}_{i,i}(u_i, a, \nu, \varepsilon)}{\varepsilon} = \alpha_i,$$

Functions $\tilde{I}_{i,j}$ are ε-regularly smooth in (u, a, ν) and :

$$\tilde{I}_{i,j}(u, 0, \nu, 0) = I_{i,j}(u, \nu)$$

Here $\alpha = (\alpha_1, \dots, \alpha_k)$ are ε-regularly smooth functions of the parameter λ, such that α_i **is equivalent to the breaking parameter** a_i **of the** i^{th} **canard transition** (if it is a jump transition) or **equivalent to the rescaled breaking parameter** $\frac{a_i}{\varepsilon}$, **of** i^{th} **canard transition** (if it is a Hopf transition). **Up to now, we can consider and we will assume that** $\lambda = (\alpha, \nu)$.

The problem :

Let $\Gamma_{\nu_0}(u_0)$ be a canard cycle.
Compute or estimate the cyclicity $\mathrm{Cycl}\Big(X_{\lambda,\varepsilon}, \Gamma_{\nu_0}(u_0)\Big)$ of this canard cycle in its slow-fast family $X_{\lambda,\varepsilon}$.

In the sequence of the talk I give some results obtained in [1]–[5].

Case $k = 1$ ([1],[2])

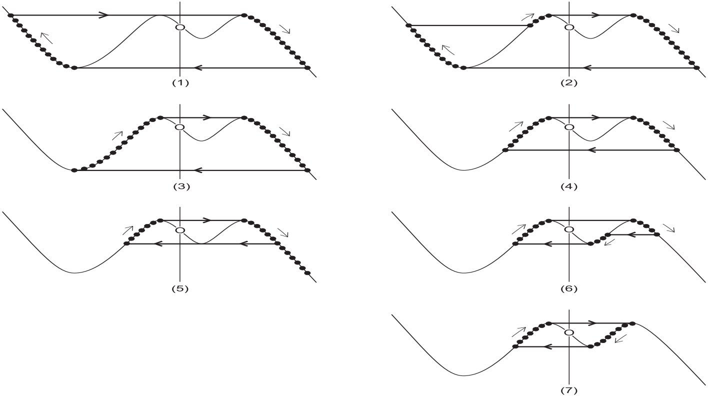


Figure from [2] : the cases (1), (3), (5), (7) are transitory canard cycles, not considered in this talk.

We have a single equation in a layer 1-dim variable u :

$$\Delta(u,\alpha,\nu,\varepsilon) = \exp\frac{\tilde{I}(u,\alpha,\nu,\varepsilon)}{\varepsilon} - \exp\frac{\tilde{J}(u,\alpha,\nu,\varepsilon)}{\varepsilon} = \alpha,$$

where $\tilde{I}(u,0,\nu,0) - \tilde{J}(u,0,\nu,0) = I(u,\nu) - J(u,\nu)$ is the slow divergence integral of the canard cycle $\Gamma_\nu(u)$.

It is easy to show that the **derivate equation** $\{\frac{\partial \Delta}{\partial u} = 0\}$ is smoothly equivalent, when $\varepsilon > 0$, to an equation :

$$I(u,\nu) - J(u,\nu) + o_\varepsilon(1) = 0,$$

with a remainder ε-regularly smooth in (u,λ).

From this, we deduce for instance that :

If the smooth function $I(u,\nu_0) - J(u,\nu_0)$ has a zero of multiplicity l at $u = u_0$, then :

$$\mathrm{Cycl}\Big(X_{\lambda,\varepsilon}, \Gamma_{\nu_0}(u_0)\Big) \le l + 1$$

Case $k = 2$ ([3],[5])

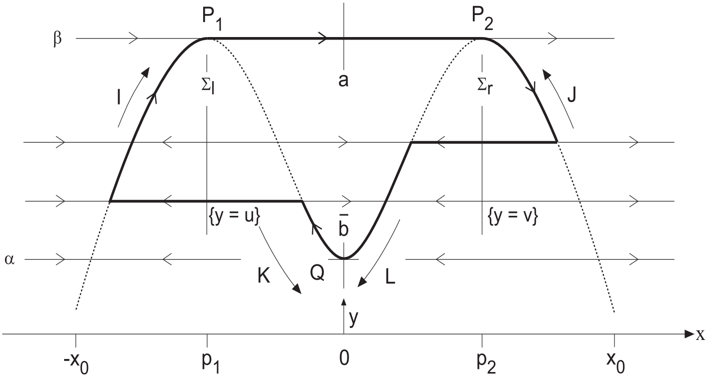


Figure 4: Canard cycles with two breaking and phase parameters

Figure from [3], with one jump and one Hopf transition

We have a system of two equations in a 2-dim layer variable $u=(u_1,u_2)$:

$$\begin{cases} \exp(\frac{\tilde{I}(u_1,\lambda,\varepsilon)}{\varepsilon}) - \exp(\frac{\tilde{J}(u_2,\lambda,\varepsilon)}{\varepsilon}) = \alpha_1 \\ \exp(\frac{\tilde{K}(u_1,\lambda,\varepsilon)}{\varepsilon}) - \exp(\frac{\tilde{L}(u_2,\lambda,\varepsilon)}{\varepsilon}) = \alpha_2, \end{cases}$$

with breaking parameter $\alpha=(\alpha_1,\alpha_2)$ and $\lambda=(\alpha,\nu)$. The functions $\tilde{I},\ldots$ are ε-regularly smooth in (u_1,u_2,λ). We have 4 **slow divergence integrals** : $I(u_1,\nu), K(u_1,\nu),\ J(u_2,\nu)$ and $L(u_2,\nu)$. $\tilde{I}(u_1,0,\nu,0)=I(u_{1,n}\,u)$ and so on.

Using Khovanskii ideas, we prove in instance in [5] that :

If the two smooth curves

$\{I(u_1,\nu)-J(u_2,\nu)=0\}$ **and**

$\{K(u_1,\nu)-L(u_2,\nu)=0\}$ **have a contact of**

finite order l, **at a point** $u^0=(u_1^0,u_2^0)$ **for**

$\nu=\nu_0$, **then**

$$\mathrm{Cycl}\Big(X_{\lambda,\varepsilon},\Gamma_{\nu_0}(u_0)\Big)\leq l+2.$$

General case $k \geq 3$ ([4],[6])

It is no longer possible to reduce the question to a system of equations based on the slow fast integral. We have to introduce **two restrictions** on the canard cycle $\Gamma_{\nu_0}(u^0)$ $(u^0 = (u_1^0, \dots, u_k^0))$:

1. $$\prod_{i=1}^{n} I'_{i,i}(u_i^0, \nu_0) \neq \prod_{i=1}^{n} I'_{i,i-1}(u_{i-1}^0, \nu_0). \qquad (1)$$

   We say that the canard cycle is **well-balanced.**

2. We restrict to a **rescaled layer** defined by :

$$u_i = u_i^0 + \varepsilon U_i, \quad \text{for} \quad i = 1, \dots, k,$$

$U = (U_1, \dots, U_k)$ is restricted to a compact domain.

Equation for limit cycles bifurcating from a well-balanced canard cycle $\Gamma_{\nu_0}(u_0)$ in a *rescaled layer* are the fixed points of the return map, which has the form :

$$P_{\lambda,\varepsilon}(\xi) \ : \xi \to \phi^r_{\bar{\alpha}}(\xi) + o_\varepsilon(1),$$

where the reminder $o_\varepsilon(1)$ is ε-regularly smooth in the other variables.

The function $\phi^r_{\bar\alpha}(\xi)$ is a **composition :**

$$\phi^r_{\bar\alpha}(\xi) = \phi^{r_k}_{\bar\alpha_k} \circ \phi^{r_{k-1}}_{\bar\alpha_{k-1}} \circ \ldots \circ \phi^{r_1}_{\alpha_1}(\xi),$$

with $r = (r_1, \ldots, r_k)$ and $\bar\alpha = (\bar\alpha_1, \ldots, \bar\alpha_k)$.

The $\phi^{r_i}_{\alpha_i}$ are **translated power functions** given by :

$$\phi^{r_i}_{\bar\alpha_i}(\xi) = \bar\alpha_i + \xi^{r_i}, i = 1, \ldots, k.$$

In the above formulas, we have that :

1. $r_i = I'_{i,i}(u_i^0, \nu_0)/I'_{i,i-1}(u_{i-1}^0, \nu_0)$, for $i = 1, \ldots, k$,

2. $\bar{\alpha}_i = \exp \frac{|I_{i,i}(u_i^0)|}{\varepsilon} \cdot \alpha_i$ for $i = 1, \ldots, k$, is obtained by rescaling α_i by the factor $\exp \frac{|I_{i,i}(u_i^0)|}{\varepsilon}$.

3. The variable ξ is some power of U_1.

The fixed points are to be searched for ξ in an arbitrarily in an arbitrarily large interval I in $\mathbb{R}^+$ (which defines a rescaled layer) and for the parameter $\bar{\alpha} \in A$ and $r \in R$, where A, R are arbitrarily large compact domains.
The well-balanced condition is needed to obtain the above form of $P_{\lambda,\varepsilon}$ with $\Pi_i r_i = r_1 \cdots r_k \neq 1$.

By the **Khovanskii Theory of fewnomials,** it is known that there exists a universal finite bound $M(k)$ for the number of **isolated fixed points** in the function family $\phi^r_{\bar{\alpha}}(\xi)$, **without restriction** on the variable $\xi \in \mathbb{R}^+$ or the parameter $(r, \bar{\alpha})$.

There exists an open dense set $\mathcal{R}$ of generic values of r, such that all fixed points of $\phi_\alpha^{\mathbf{r}}(\xi)$ have a finite multiplicity.

Then, for $r \in \mathcal{R}$, the function $\phi_\alpha^{\mathbf{r}}(\xi)$ has less than $M(k)$ fixed points, counted with their multiplicity (c.w.t.m).

Let us chose a connected compact domain $R \subset \mathcal{R}$ and an arbitrarily interval I for ξ and compact domain A for $\bar{\alpha}$.
Then, the return map $P_{\lambda,\varepsilon}(\xi)$ has also less than $M(k)$ fixed point s in I for $(r, \alpha) \in R \times A$, **if ε is small enough.** All these points have a finite multiplicity. **The cyclicity of $\Gamma_{\nu_0}(u_0)$ inside a rescaled layer is bounded by $M(k)$.**

What is known about M(k) ?

For $k = 1$, we have that
$\phi^{r_1}_{\bar{\alpha}_1}(\xi) = \xi^{r_1} + \bar{\alpha}_1$, with $r_1 \neq 1$.
Then :

$$M(1) = 2.$$

A well-balanced canard cycle, **with slow divergence integral zero** and with one breaking parameter is just a **generic double canard cycle** (as defined in [1]) and its cyclicity is 2, corresponding to a fold bifurcation.

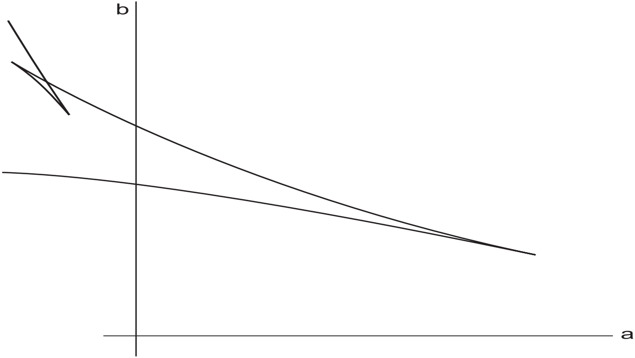
For $k = 2,$ the study is a little more involved and there are different generic situations. Anyway it is rather easy to prove that :

$$M(2) = 3.$$

The more complicated situation is a cusp bifurcation. A well-balanced canard cycle with two breaking parameters corresponds to the case where the curves $\{I - J = 0\}$ and $\{K - L = 0\}$ are transverse and we haave seen that the cyclicity is less than $1 + 2 = 3.$

Then, for $k = 1, 2$ we obtain the bound expected for an elementary catastrophe (fold resp. cusp catastrophes), although the catastrophe theory does not apply here.
The following example shows that it is not the case for $k = 3$.

For $k = 3$ an example by Panazzolo, given in [4], exhibits a bifurcation with 3 cusp points on some generic sections of the parameter space (one has to choose particular values of (r_1, r_2, r_3). **Then we have 5 fixed points and $M(3) \geq 5$.**



A section $c = \mathrm{Const}$ in the 3-dim parameter space (a, b, c)

The value of $M(k)$ is unknown for $k > 3$. Using the theory of fewnomials of Khovanskii, D. Panazzolo has obtained the following estimate :

$$M(k) \leq 2^{k(2k-1)}(k+1)^{2k}.$$

Indeed, this estimate is very rough as it gives $M(1) \leq 8$, $M(2) \leq 5184$ and $M(3) \leq 2^{27} = 134217768$.

We have proved in [6] the following :
Theorem $M(3) = 5.$

The proof is much more involved than in cases $k = 1, 2$. I want to give a hint of the proof.

The proof that $M(3) = 5$

Changing the name of rescaled breaking parameters in (α, β, γ), the return map in the rescaled layer is :

$$P_{\lambda,\varepsilon} \; : \xi \mapsto \gamma + (\beta + (\alpha + \xi^{r_1})^{r_2})^{r_3} + o_\varepsilon(1).$$

We have to search the zeros of the difference function

$$\delta_{\lambda,\varepsilon}(\xi) = P_{\lambda,\varepsilon}(\xi) - \xi$$

First step

Using ideas of Khovanskii, we can prove that the equation $\delta^1_{\lambda,\varepsilon}(\xi) = \frac{\partial \delta_{\lambda,\varepsilon}}{\partial \xi}(\xi) = 0$, is smoothly equivalent, when $\varepsilon > 0$, to an equation of the form :

$$\xi^\sigma - G^{-1}\xi^{\sigma_1}(\xi - \alpha)^{\sigma_2} - \beta + o_1(\varepsilon) = 0,$$

where $G = r_1 r_2 r_3 \neq 1$ and the exponants $\sigma, \sigma_1, \sigma_2$ are rational functions of (r_1, r_2, r_3).

By using Rolle's Theorem, if $\delta^1_{\lambda,\varepsilon}$ has l roots, counted with their multiplicity (c.w.t.m), then $\delta_{\lambda,\varepsilon}$ has less than $l+1$ roots, c.w.t.m.

A trick to reduce $\delta^1_{\lambda,\varepsilon}(\xi)$ to a linear combination of parameters

I just want to explain the case $\alpha > 0$. The other cases $\alpha = 0$ and $\alpha < 0$ can be treated similarly. We introduce the variable μ by $\xi = \alpha(1+\mu)$ with $\mu > 0$ (since $\alpha\mu = \xi - \alpha > 0$). Then $\varphi = \delta^1$ transforms into :

$$\varphi_+(\mu) = \alpha^\sigma(1+\mu)^\sigma - G^{-1}\alpha^{\sigma_1+\sigma_2}\mu^{\sigma_2}(1+\mu)^{\sigma_1} - \beta.$$

The expression of $\varphi_+(\mu)$ is linear in the parameter functions :

$$\alpha^{\sigma}, \quad -G^{-1}\alpha^{\sigma_1+\sigma_2} \quad \text{and} \quad -\beta.$$

Digression on the division-derivation procedure

If we consider a smooth function family, linear in the $n+1$ parameter functions $(a_1(\lambda),\ldots,a_{n+1}(\lambda))$:

$$f(z)=a_1(\lambda)g_1(z,\lambda)+\cdots+a_{n+1}(\lambda)g_{n+1}(z,\lambda),$$

Under some mild conditions, it is possible to eliminate the parameters one by one the following procedure of division-derivation :

Let us suppose that $a_{n+1} \neq 0$ (otherwise, we can work with a shorter combination). Dividing by a_{n+1} we can assume that $a_{n+1} = 1$.

First, we **divide** by g_1, if possible and next, **derive** in z. We obtain a new function $f^1(z)$ which is linear on the n parameter functions $(a_2, \ldots, a_{n+1})$.

If it is possible to iterate this procedure n times, we obtain at the end a function $f^n(z)$, **independant of the parameter** and smoothly **equivalent to the n^{th} derivative $\frac{\partial^n}{\partial z^n} f$. If $f^n(z)$ has l zeros, c.w.t.m, then f has $n+l$ zeros, c.w.t.m.**

End of digression

We can apply the procedure of division-derivation to φ_+, which is linear in parameter functions. After two steps and after dividing by $\alpha > 0$, we obtain a polynomial of degree ≤ 2 in μ :

$$\varphi_+^2(\mu) = (\sigma_1 + \sigma_2)(\sigma_1 + \sigma_2 - \sigma)\mu^2 + \sigma_2(2\sigma_1 + 2\sigma_2 - \sigma - 1)\mu + \sigma_2(\sigma_2 - 1).$$

A consequence of $G \neq 0$, i.e. that the canard cycle is well-balanced is that φ_+^2 is a **non trivial polynomial :** one the three coefficients of φ_+^2 is non-zero.

We have passed from $\delta_{\lambda,\varepsilon}$ to a function $\delta^3_{\lambda,\varepsilon} = \varphi_+^2 + o_\varepsilon(1)$ by three steps of the procedure of division-derivation.

As the function φ_+^2 **has less than 2 zeros, c.w.t.m,** it is the same for the function $\delta_{\lambda,\varepsilon}^3$, **for ε small enough.** Then $\delta_{\lambda,\varepsilon}$ **itself has less than 5 zeros, c.w.t.m.**

As a consequence we have that $M(3) \leq 5$. Taking into account the example of D. Panazzolo we obtain that $M(3) = 5$

Open questions

(1) Find a more accurate value of $M(k)$.

(2) Prove that $M(k)$ has a large minorant in terms of k (exponential ?)

(3) Compute the cyclicity of a well-balanced for $k \geq 3$ (and not just the cyclicity restricted to a rescaled layer).

(3) Prove that the cyclicity is bounded by $M(k)$: i.e., configurations with a maximum number of bifurcating limit cycles must occured for some parameter value into a rescaled layer.

(5) And what about the canard cycle which are not well-balanced ? (a general result, as recalled, was obtained only for $k = 1, 2$).

THANKS YOU

FOR YOUR ATTENTION !